Problem Set 2

MATH 778C, Spring 2009, Austin Mohr (with John Boozer)

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1. Let $S \subset \mathbb{Z}_p^\times$ be such that $S = -S$, where p is prime. Define the "circulant graph" G_S by $V(G_S) = \mathbb{Z}_p$ and $\{x, y\} \in E(G_S)$ iff there exists $s \in S$ so that $x + s = y$. Define the k^{th} discrete Fourier coefficient of a function $f : \mathbb{Z}_p \to \mathbb{C}$, for $k \in \mathbb{Z}_p$, by

$$\hat{f}(k) = \frac{1}{\sqrt{p}} \sum_{j=0}^{p-1} e^{2\pi ijk/p} f(j).$$

Show that

$$\lambda(G) = \frac{\sqrt{p}}{|S|} \max_{k \neq 0} |\hat{\chi}_S(k)|,$$

where χ_S is the characteristic function of S. Hint: First prove that $\{f_k\}_{k \in \mathbb{Z}_p}$ is an orthonormal basis of $\mathbb{C}^{\mathbb{Z}_p}$, where $f_k(j) = \exp(2\pi ijk/p)/\sqrt{p}$.

Proof.

$$< f_k, f_k >= \sum_{j=0}^{p-1} \frac{e^{2\pi ijk/p}}{\sqrt{p}} \frac{e^{-2\pi ijk/p}}{\sqrt{p}} = \sum_{j=0}^{p-1} \frac{1}{p} = 1$$

Let $k \neq l$

$$< f_k, f_l >= \sum_{j=0}^{p-1} \frac{e^{2\pi ijk/p}}{\sqrt{p}} \frac{e^{-2\pi ijl/p}}{\sqrt{p}} = \sum_{j=0}^{p-1} \frac{e^{2\pi ij(k-l)/p}}{p} = 0$$

So $\{f_k\}_{k \in \mathbb{Z}_p}$ is an orthonormal basis.

Let A be the random walk matrix of G_S. $A_{l,j} = \frac{\chi_S(l-j)}{|S|}$ Then the jth entry of Af_k is

$$\begin{aligned} \frac{1}{|S|} \sum_{l=0}^{p-1} \chi_S(l-j) \frac{e^{2\pi ilk/p}}{\sqrt{p}} &= \frac{e^{2\pi ijk/p}}{\sqrt{p}} \frac{1}{|S|} \sum_{l=0}^{p-1} \chi_S(l-j) e^{2\pi i(l-j)k/p} \\ &= \frac{\sqrt{p}}{|S|} \hat{\chi}_S(k) \frac{e^{2\pi ijk/p}}{\sqrt{p}} \end{aligned}$$

So f_k is an eigenvector with the eigenvalue $\frac{\sqrt{p}}{|S|}\hat{\chi}_S(k)$. Since $\lambda(G)$ is the second largest eigenvalue, with the largest occurring with f_0, then taking the maximum of the eigenvalues of $f_1, \ldots, f_{p-1}$ gives us $\lambda(G)$. Note that since f_k is a basis of eigenvectors, we are assured of getting all of the eigenvalues. □

2. Let A and B be symmetric stochastic matrices. Prove that $\lambda(A+B) \leq \lambda(A) + \lambda(B)$.

Proof. Let $\mathbf{w}$ be the vector that maximizes $||(A+B)\mathbf{v}||_2$ over all vectors $\mathbf{v} \in \mathbf{1}^{\perp}$.

$$\begin{aligned}\lambda(A+B) &= ||(A+B)\mathbf{w}||_2 \\ &= ||A\mathbf{w} + B\mathbf{w}||_2 \\ &\leq ||A\mathbf{w}||_2 + ||B\mathbf{w}||_2 \text{ (by the triangle inequality)}\end{aligned}$$

It is clear that $||A\mathbf{w}||_2 \leq \max_{\mathbf{v}\in\mathbf{1}^{\perp}} ||A\mathbf{v}||_2$ and $||B\mathbf{w}||_2 \leq \max_{\mathbf{v}\in\mathbf{1}^{\perp}} ||B\mathbf{v}||_2$, hence

$$\begin{aligned}\lambda(A+B) &\leq ||A\mathbf{w}||_2 + ||B\mathbf{w}||_2 \\ &\leq \max_{\mathbf{v}\in\mathbf{1}^{\perp}} ||A\mathbf{v}||_2 + \max_{\mathbf{v}\in\mathbf{1}^{\perp}} ||B\mathbf{v}||_2 \\ &= \lambda(A) + \lambda(B)\end{aligned}$$

□

3. Prove that, for every n-vertex d-regular graph, there is some subset S of $n/2$ vertices so that $|E(S, \overline{S})| \leq dn/4 + O(1)$. Conclude that no (n, d, ρ)-edge expander family exists if $\rho > 1/2$.

Proof. Let S be a random subset of $\frac{n}{2}$ vertices. The probability of an edge belonging to $E(S, \overline{S})$ is the same as the probability that its endpoints belong to different subsets (with respect to S and $\overline{S}$). Now, the presence of loops in G can only lower this probability, making the desired upper bound easier to achieve. Assume, then, that G does not contain loops. Fix a vertex v in G. Each of its d neighbors has probability $\frac{\frac{n}{2}}{n-1} = \frac{1}{2} + \frac{1}{2(n-1)}$ of belonging to the opposite subset (since there are no loops, we know that v is not a neighbor of itself, hence the denominator of $n-1$). We see that v contributes an average of $d(\frac{1}{2} + \frac{1}{2(n-1)})$ edges to $E(S, \overline{S})$. Taken over all vertices of G (with a factor of $\frac{1}{2}$ to account for the double-counting of edges), the expected value of $|E(S, \overline{S})|$ is

$$\begin{aligned}\frac{n}{2} \cdot d\left(\frac{1}{2} + \frac{1}{2(n-1)}\right) &= \frac{nd}{4} + \frac{nd}{4(n-1)} \\ &= \frac{nd}{4} + O(1)\end{aligned}$$

Therefore, there must be some subset S such that $|E(S,\overline{S})| \leq \frac{nd}{4} + O(1)$.

Now, suppose $\rho > \frac{1}{2}$. A family of graphs $\{G_n\}$ is an (n,d,ρ)-edge expander family if, for each G_n, any subset S of at most $\frac{n}{2}$ vertices has the property that $|E(S,\overline{S})| \geq \rho d|S|$. If we let $|S| = \frac{n}{2}$ and use the given ρ, we require that $|E(S,\overline{S})| > \frac{nd}{4}$. By the first part of the problem, we know that, for large n, we can find some S that breaks this bound. Hence, there is no edge expander family for $\rho > \frac{1}{2}$. □

4. Let G be an (n,D,ρ)-edge expander and G' be a (D,d,ρ')-edge expander, for $\rho,\rho' > 0$. Prove that $G \textcircled{R} G'$ is a $(nD, 2d, \rho^2\rho'/80)$-edge expander.

Proof. Let $H = G \textcircled{R} G'$ and let S be a subset of $V(H)$ of at most $\frac{nD}{2}$ vertices. We can view H as being made up of n D-vertex clusters, and so we can partition the vertices in S based on the cluster to which they belong. To establish the claim, we must show that there are $\frac{1}{80}\rho^2\rho' \cdot 2d|S|$ edges leaving S.

Define S_i to be the set of vertices of S belonging to cluster i. We partition the indices $1,\ldots,n$ of the clusters into two sets I' and I'' as follows: If $|S_i| \leq (1 - \frac{1}{4}\rho)D$, put i in I'. Otherwise, put i in I''. Finally, define $S' = \bigcup_{i \in I'} S_i$ and $S'' = \bigcup_{i \in I''} S_i$. Observe that $S_\prime \cup S_{\prime\prime} = S$.

Case $|S'| \geq \frac{1}{10}\rho|S|$

Since G' is a (D,d,ρ')-edge expander, there are at least $\frac{1}{4}\rho\rho' d|S_i|$ edges leaving S_i within each cluster. Taken over all clusters, we see there are $\frac{1}{4}\rho\rho' d|S'|$ edges leaving $S' \subset S$. Since $|S'| \geq \frac{1}{10}\rho|S|$, we conclude that there are at least $\frac{1}{80}\rho^2\rho' \cdot 2d|X|$ edges leaving S in H.

Case $|S'| \leq \frac{1}{10}\rho|S|$

We see immediately that $|S''| > (1 - \frac{1}{10}\rho)|S|$. For $i \in I''$, $|S_i| \geq (1 - \frac{1}{4}\rho)D$, and so $\frac{|S''|}{D} \leq |I''| \leq \frac{|S''|}{(1-\frac{1}{4}\rho)D}$. Furthermore, since $|S''| \leq \frac{1}{2}nD$, we see that $|I''| \leq \frac{2}{3}n$. Since G is an (n,D,ρ)-edge expander, we can find a set of edges M with $|M| \geq \frac{1}{2}\rho D|I''|$ between the vertices of I' and I'' (that is, the sets X_i with i belonging to I' and I'', respectively). Now, there are $d|M| \geq \frac{1}{\rho} dD|I''|$ edges in H connecting vertices in the cycles whose index belongs to I' with vertices in the clusters whose index belongs to I''. For $i \in I''$, $|S_i| \geq (1 - \frac{1}{4}\rho)D$. Hence, at most $\frac{1}{4}\rho dD|I''|$ of the $d|M|$ edges connect a vertex not in S_i with a vertex in one of the clusters whose index belongs to I'. So, for $i \in I''$, at least $\frac{1}{4}\rho dD|I''|$ edges connect vertices of the S_i with vertices of the aforementioned clusters. The number of edges connecting the remaining clusters (having $i \in I''$) with S' is at most $d|S'|$. Since $|S'| \leq \frac{1}{10}\rho|S| \leq \frac{1}{6}\rho D|I''|$, there are at most $\frac{1}{6}\rho dD|I''|$ corresponding edges. Hence, at least $\frac{1}{12}\rho d D|I''|$ edges connect vertices of the S_i ($i \in I''$) with the complement of S_j ($j \in I'$) inside the cycles with index j. Since $|I''| \geq \frac{|S''|}{D}$ and $|S''| \geq \frac{1}{2}|S|$, we conclude that there at least $\frac{1}{48}\rho \cdot 2d|S|$ edges leaving S, proving the claim. □

5. A (countably) infinite locally finite graph G is said to be an f-expander, for $f : \mathbb{N} \to \mathbb{R}^+$, if

$$\min_{S: n \leq |S| < \infty} |E(S, \overline{S})| = \Theta(f(n))$$

Give an example of a 1-expander, an $n^{2/3}$-expander and an n-expander. (Of course, you need to provide proofs that they work.)

Claim. *Define P to be the path beginning at a root r and extending infinitely in one direction. P is a 1-expander.*

Proof. Since the graph is infinite and connected, $|E(S, \overline{S})| \geq 1$ for any S. Let $|S| = n$. If we construct S by selecting the n vertices closest to r (including r itself), we see that $|E(S, \overline{S})| = 1 = \Theta(1)$. Hence, P is a 1-expander. □

Claim. *Define T to be an infinite binary tree with root r. T is an n-expander.*

Proof. Suppose first that the graph induced by S (call it T') is disconnected. Define the e-degree of a vertex (subgraph) to be the number of edges that vertex (subgraph) contributes to $E(S, \overline{S})$. By translating this lowest component of T' upward so that it becomes connected with another component, we can only reduce the e-degree of both components. Hence, the S which gives the minimum value of $|E(S, \overline{S})|$ must induce a connected subgraph.

Observe next that if we translate T' so that it is rooted at r (that is, we rechoose S so that we have a new tree that is isomorphic to T' but rooted at r), then $|E(S, \overline{S})|$ can only be reduced. This is because the number of edges in $E(S, \overline{S})$ that extend *below* the tree remains unchanged, while some edges in $E(S, \overline{S})$ that extended *above* the tree have been removed during the tranlsation. Hence, the overall e-degree of T' is reduced. We will assume, then, that T' is rooted at r.

Now, let $|S| = n$. We observe next that $|E(S, \overline{S})| = n + 1$. To see this, build T' by first choosing r and successively appending edges as prescribed by S. When we have only r, $|E(S, \overline{S})| = 2$. At the next stage, we append an leaf to r. This decreases the e-degree of r by 1, but the leaf itself has e-degree of 2, resulting in a net gain of one edge in $E(S, \overline{S})$. Continuing in this way, we see that $|E(S, \overline{S}| = n + 1 = \Theta(n)$. □

Claim. *Define T to be the infinite three-dimensional lattice. T is an $n^{\frac{2}{3}}$-expander.*

Proof. The minimization of $|E(S, \overline{S})|$ is equivalent to the minimization of the "surface area" (i.e. the number of vertices of S exposed to vertices of $\overline{S}$), which occurs when the graph induced by S forms a cube. Let $|S| = n$ with $n = k^3$. The number of vertices on the surface is around $6k^2 = 6n^{\frac{2}{3}} = \Theta(n^{\frac{2}{3}})$. □