Problem Set 3

MATH 777, Spring 2010, Mohr

April 20, 2010

1 Problem 2

Theorem 1.1. *(Erdős-Szekeres-Klein) For every k, there is an n so that any n points in the plane with no three collinear contain the vertices of a convex k-gon.*

Proof. We first define a coloring of $\binom{[n]}{3}$ (the value of n will be determined later). Given a 3-subset $\{x, y, z\}$ with $x < y < z$, color the subset "clockwise" if passing from x through y to z via line segments results in a clockwise motion. Otherwise, color the subset "counterclockwise" (these colors suffice, as no three points are collinear). By Ramsey, take n large enough to ensure a monochromatic k-subset S of $[n]$ (without loss of generality, let it be colored "clockwise").

Let X denote the convex hull of S. We claim that every point of S lies on the boundary of X, thus establishing the theorem. Suppose, to the contrary, that some element $a \in S$ lies on the interior of X. Let b denote the successor in S of a and let c be any other element of S. Divide the plane into two half-planes by the line passing through a and b. Now, if $c > b$, c must lie in a particular half-plane (call it the right half-plane), since the motion from a through b to c must be clockwise. Similarly, if $c < b$, then $c < a$, and the clockwise motion from c through a to b also places c in the right half-plane. Since c was arbitrary, we conclude that the left half-plane is empty, and so a lies on the boundary of X. Similarly, the fact that a was chosen arbitrarily implies that all points in S lie on the boundary of X, and the theorem follows. □

2 Problem 3

Proposition 2.1. *Show that the "exceptional pairs" part of the Szemerédi Regularity Lemma is unavoidable. In other words, show that the following* *statement is false:* $\forall \epsilon > 0, m > 0, \exists N, M$ *so that, if* G *is a graph on* $n \geq N$ *vertices,* $\exists k$ *and there exists a partition* $V_0, \ldots, V_k$ *of* $V(G)$ *into blocks so that* $|V_i| = |V_j|$ *for all* $i, j > 0$*,* $|V_0| \leq \epsilon n$*,* all *pairs* (V_i, V_j) *with* $i \neq j$ *are* ϵ*-regular, and* $m \leq k \leq M$*.*

Proof. (Idea) Let G be the half-graph and denote $V(G)$ by $[2]\times[n]$. Partition the $V(G)$ into $V_1, \ldots, V_k$ all of equal size and suppose that they are all ϵ-regular. For small ϵ, one can find V_i and V_j with

1. $|V_i \cap (\{1\} \times [n])|$ and $|V_j \cap (\{2\} \times [n])|$ large compared to $\epsilon|V_i| = \epsilon|V_j|$, and

2. V_i and V_j have significant "overlap" (that is, there are many vertices of $|V_i|$ that are lower than the highest vertices of $|V_j|$, or vice versa)

With these V_i and V_j in hand, one can bound $d(V_i, V_j)$, but demonstrate areas of either zero density (if V_i is generally higher than V_j) or high density (if V_i is generally lower than V_j), contrary to our assumption that all pairs are ϵ-regular. □

3 Problem 4

Proposition 3.1. *If* G *is a graph with no isolated vertices, then*

$$R(P^2, G) = \begin{cases} |G| & \text{if } \overline{G} \text{ has a perfect matching} \\ 2|G| - 2\alpha(L(\overline{G})) - 1 & \text{otherwise.} \end{cases}$$

(The quantity $\alpha(L(H))$ *is called the* <u>matching number</u> *of* H *and is often denoted* $\nu(H)$*.)*

Proof. Consider first the case where $\overline{G}$ has a perfect matching and let n denote $|G|$. Since $\overline{G}$ has a perfect matching, we can partition $V(G)$ into distinct unordered pairs $\{v_i, w_i\}$ such that, for all i, v_i is not adjacent to w_i.

Evidently, $R(P^2, G) \geq n$. We show now that every graph H on n vertices with $P^2 \not\subseteq H$ has $G \subseteq \overline{H}$. Since $P^2 \not\subseteq H$, H has no incident edges. Hence we may assume, without loss of generality, that H is a collection of $\frac{n}{2}$ disjoint edges. Now, observe that $\overline{H}$ is the complete graph on n vertices with this collection of $\frac{n}{2}$ edges removed. By our observation about $V(G)$, the endpoints of these removed edges correspond precisely to the $\frac{n}{2}$ nonadjacent vertices in G, and so $G \subset \overline{H}$, as desired.

Consider now the case where $\overline{G}$ does not have a perfect matching. Let k denote the size of a maximum matching in $\overline{G}$. Since the matching is not perfect, there are k unordered pairs of nonadjacent vertices $\{v_i, w_i\}$ with $v_i, w_i \in V(G)$, but this does not comprise all of $V(G)$. Observe also that the induced subgraph $G[V(G) \setminus \bigcup_{i=1}^{k}\{v_i, w_i\}]$ is a clique of size $n - 2k$ (if not, then k is not the size of the largest clique in $\overline{G}$). Let now H be a graph on $2(n - k - 1)$ vertices with $P^2 \nsubseteq H$. As before, we assume, without loss of generality, that H is a collection of $n - k - 1$ disjoint edges. Now, k of these edges account for the k unmatched edges in $\overline{H}$ as required by G. This leaves $n-2k-1$ disjoint edges in H, but this is too small to accomodate an $(n-2k)$-clique in $\overline{H}$, which is also required by G. Hence, $R(P^2, G) < 2(n - k - 1)$. By this same construction, however, we see that adding even a single vertex (which must necessarily be disjoint from the rest of H since $P^2 \nsubseteq H$) does allow for the required $(n - 2k)$-clique. Hence, $R(P^2, G) = 2n - 2k - 1$, as desired. □

4 Problem 5

Proposition 4.1. *For every $k \in \mathbb{N}$, there is an $n \in \mathbb{N}$ such that, for every partition of $\{1, \ldots, n\}$ into k sets, at least one of the subsets contains numbers x, y, z such that $x + y = z$.*

Proof. Let k be given and let $c : [n] \to [k]$ be any k-coloring of $[n]$ (the value of n will be determined later). Define now $f : \binom{[n]}{2} \to [k]$ via $f(\{i, j\}) = c(|i - j|)$. By Ramsey, we will take n to be large enough so that f admits a monochromatic 3-subset $\{a, b, c\}$ of $\binom{[n]}{2}$. Without loss of generality, suppose $a < b < c$. Let now $x = b - a$, $y = c - b$, and $z = c - a$. Since $\{a, b, c\}$ is monochromatic under f, we have $f(\{a, b\}) = f(\{b, c\}) = f(\{a, c\})$, but this is precisely $c(|b - a|) = c(|b - c|) = c(|a - c|)$. Hence, $c(x) = c(y) = c(z)$, and so x, y, and z indeed belong to the same subset of $[n]$ under c. Finally,

$$\begin{aligned} x + y &= (b - a) + (c - b) \\ &= c - a \\ &= z, \end{aligned}$$

as desired. □

5 Problem 6

Let $(X, \leq)$ be a totally ordered set. Define a graph G by letting $V(G) = \binom{X}{2}$, and

$$E(G) = \left\{ \{\{x, y\}, \{y, z\}\} \in \binom{V(G)}{2} : x < y < z \right\}.$$

Proposition 5.1. *The graph G contains no triangle.*

Proof. Let $\{x_1, y_1\}$, $\{x_2, y_2\}$, and $\{x_3, y_3\}$ be vertices of G. Without loss of generality, let $\{x_1, y_1\}$ be adjacent to $\{x_2, y_2\}$ by satisfying $x_1 < y_1 = x_2 < y_2$. We investigate the possiblity that $\{x_2, y_2\}$ is adjacent to $\{x_3, y_3\}$ and $\{x_3, y_3\}$ is adjacent to $\{x_1, y_1\}$ simultaneously by considering four cases.

<u>Case 1</u> $x_2 < y_2 = x_3 < y_3$ and $x_1 < y_1 = x_3 < y_3$
We have the inconsistent statement $y_1 = x_2 < y_2 = x_3 = y_1$.

<u>Case 2</u> $x_2 < y_2 = x_3 < y_3$ and $x_3 < y_3 = x_1 < y_1$
We have the inconsistent statement $x_1 < y_1 = x_2 < y_2 = x_3 < y_3 = x_1$.

<u>Case 3</u> $x_3 < y_3 = x_2 < y_2$ and $x_1 < y_1 = x_3 < y_3$
We have the inconsistent statement $y_1 = x_3 < y_3 = x_2 = y_1$.

<u>Case 4</u> $x_3 < y_3 = x_2 < y_2$ and $x_3 < y_3 = x_1 < y_1$
We have the inconsistent statement $x_1 < y_1 = x_2 = y_3 = x_1$.

Arriving at a contradiction in all cases, we conclude that there can be no triangle in G. □

Proposition 5.2. *For every $k \in \mathbb{N}$, there exists an $n \in \mathbb{N}$ so that, if $|X| \geq n$, then $\chi(G) \geq k$.*

Proof. Let $k \geq 1$ be given. The generalized Rasmey theorem states that there is an n such that every n-set X has a monochromatic 3-subset with respect to any k coloring of $\binom{V}{2}$ (that is, $R(2, k, 3)$ exists). In particular, let X be a totally ordered n-set, and define G as above using this X. In the context of G, the fact that $R(2, k, 3)$ exists asserts that any k-coloring of the vertices of G admits a 3-subset $\{x, y, z\}$ of X (without loss of generality, $x < y < z$) such that the vertices $\{x, y\}$, $\{x, z\}$, and $\{y, z\}$ all receive the same color. Now, $x < y = y < z$, and so $\{x, y\}$ is adjacent to $\{y, z\}$. Hence, there is no proper k-coloring of G. Since k was chosen arbitrarily, we conclude that, for all k, there exists an n such that $\chi(G) \geq k$ whenever $|X| \geq n$, as desired. □

6 Problem 7

Definition 6.1. *A family of sets is called a <u>Δ-system</u> if every two of the sets have the same intersection.*

Proposition 6.2. *Every infinite family of sets of the same finite cardinality contains an infinite* Δ*-system.*

Proof. Let $\mathcal{F}$ be an infinite family of sets of the same finite cardinality n. Define a coloring $c : \binom{\mathcal{F}}{2} \to [n+1]$ where $c(\{X, Y\}) = |X \cap Y|$. By Ramsey, this coloring admits an infinite monochromatic subfamily $\mathcal{G}$ (colored, say, k) of $\mathcal{F}$. In the language of sets, this means that for all $X, Y \in \mathcal{G}$, $|X \cap Y| = k$. Fix now some $A \in \mathcal{G}$. Partition the infinite family $\mathcal{G} \setminus \{A\}$ into $\binom{n}{k}$ blocks where sets X and Y belong to the same block whenever $A \cap X = A \cap Y$. By the Pigeonhole Principle, one of the blocks of this partition contains an infinite subcollection $\mathcal{H}$ of $\mathcal{G}$. Now, for all $X, Y \in \mathcal{H}$, $A \cap X = A \cap Y$ and $|A \cap X| = |A \cap Y| = |X \cap Y| = k$. Hence, $(X \cap Y) \setminus A = \emptyset$, and so $X \cap Y = A \cap X = A \cap Y$. In other words, any two pair of sets contained in $\mathcal{H}$ have the same intersection, and so $\mathcal{H}$ is the desired infinite Δ-system. □

7 Problem 8

Proposition 7.1. *Let* $m, n \in \mathbb{N}$*, and assume that* $m-1$ *divides* $n-1$*. Every tree* T *of order* m *satisfies* $R(T, K_{1,n}) = m + n - 1$*.*

Proof. We show first that $R(T, K_{1,n}) \geq m + n - 1$. Consider the graph G constructed by taking $\frac{n-1}{m-1} + 1$ disjoint copies of K_{m-1}. As no connected component contains m vertices, G cannot contain T as a subgraph. To see that $\overline{G}$ contains no $K_{1,n}$, choose any vertex $v \in G$ to be the center of the star. To find a $K_{1,n}$ in $\overline{G}$, we need to find n vertices that are not adjacent to v. Now, v is adjacent to everything in its own component, which leaves only $(m-1) \cdot \frac{n-1}{m-1} = n - 1$ vertices in G. Hence, $T \not\subseteq G$ and $K_{1,n} \not\subseteq \overline{G}$, and so $R(T, K_{1,n}) > (\frac{n-1}{m-1} + 1)(m-1) = m + n - 2$ (i.e. $R(T, K_{1,n}) > (\frac{n-1}{m-1} + 1)(m-1) = m + n - 1$).

Let G now denote any graph on $m + n - 1$ vertices and suppose that $K_{1,n} \not\subseteq \overline{G}$. Avoiding a $K_{1,n}$ in $\overline{G}$ means that for any vertex $v \in G$, there are at most $n-1$ vertices *not* in the neighborhood of v. Put another way, v has at least $(m+n-2) - (n-1) = m - 1$ neighbors. As this bound holds for all vertices of G, we see that $\delta(G) \geq m - 1$. Since T is a tree on m vertices, we conclude that $T \subseteq G$, as desired. □

8 Problem 9

Proposition 8.1. *For every* $c \in \mathbb{N}$*,*

$$2^c < R(2, c, 3) \leq 3c!.$$

Proof. For ease of notation, we consider partitions of the set $[n]$.

We first establish the lower bound by induction on c.

For $c = 1$, we edge-color K^2 using one color, which certainly admits no monochromatic triangle. Hence, $2 < R(2, 1, 3)$.

Suppose now that $2^k < R(2, k, 3)$. Observe that $K^{2^{k+1}} = K^{2^k} * K^{2^k}$. By the inductive hypothesis, we can color the edges of each K^{2^k} using only k colors and avoid a monochromatic triangle. For the remaining edges (those edges connecting vertices between the K^{2^k}'s), assign the color $k + 1$. Evidently, this coloring admits no monochromatic triangle, as the edges lying between the K^{2^k}'s are of a completely different color than those within each of them. Hence, $2^{k+1} < R(2, k + 1, 3)$.

Next, we establish the upper bound, again by induction on c.

For $c = 1$, observe that a 1-edge-coloring of K^3 must admit a monochromatic triangle. Hence, $R(2, 1, 3) \leq 3$.

Suppose now that $R(2, k, 3) \leq 3k!$. Let v be a vertex of $K^{3(k+1)!}$. Observe that there are $3(k + 1)! - 1$ edges incident with v and they are colored using $k + 1$ colors. By the Pigeonhole Principle, some color (say, red) is used on at least $3k!$ of these edges. Let G be the induced subgraph on all vertices that are endpoints of these red edges (this includes v), and let H be the induced subgraph on these same vertices with the exception of v. We consider two cases. If the edges of H are colored using only k colors, then the induction hypothesis guarntees that H contains a monochromatic triangle (since $|H| \geq 3k!$). Otherwise, the edges of H make use of all $k + 1$ colors. In particular, there is some edge uw that is colored red, and so the cycle $vuwv$ is a monochromatic triangle in G. In either case, we conclude that $R(2, k + 1, 3) \leq 3(k + 1)!$. □

Proposition 8.2.

$$R(2, 3, 3) = 17$$

Proof. Observe first that $R(2, 3, 3) > 16$, since there exists a 3-edge-coloring of K^{16} containing no monochromatic triangle (see figure). Consider now any 3-edge-coloring of K^n for $n \geq 17$ in which we attempt (unsuccessfully) to avoid a monochromatic triangle. For any fixed vertex $v \in V(K^n)$, v has $n - 1$ incident edges, which is at least 16. By the Pigeonhole Principle, one of the color classes (say, red) contains at least 6 edges. Denote by U the subset of V containing the red neighbors of v and consider the induced subgraph H of K^n on U. It cannot be that H contains a red edge uw, as this would form the red triangle $vuwv$. Hence, the edges of U are colored with two colors. However, $|H| \geq R(3) = 6$, and so any 2-edge-coloring of

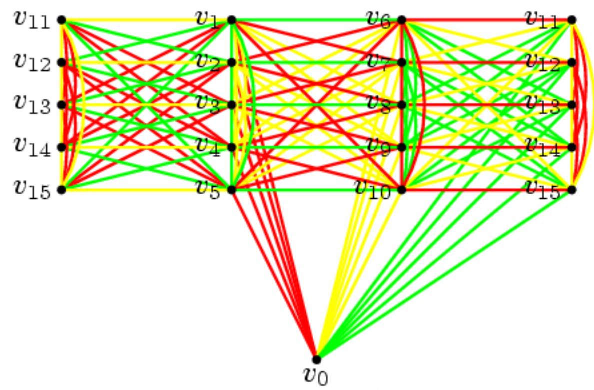

Figure 1: A 3-edge-coloring of K^{16} having no monochromatic triangle.

H must admit a monochromatic triangle. We conclude, therefore, that no 3-edge-coloring of K^n for $n \geq 17$ can avoid a monochromatic triangle, and so $R(2, 3, 3) = 17$.

□

9 Problem 10

Proposition 9.1. *Given any two graphs H_1 and H_2, there exists a graph $G = G(H_1, H_2)$ such that, for every vertex-coloring of G with colors 1 and 2, G has either H_1 colored 1 or H_2 colored 2 as an induced subgraph.*

Proof. We claim that the desired graph is $G = H_1[V(H_1) \to H_2]$. For ease of exposition, let $V(H_1) = [n_1]$ and $V(H_2) = [n_2]$, so that $V(G) = [n_1] \times [n_2]$. For each $k \in [n_1]$, we call the subset $\{k\} \times [n_2]$ of $V(G)$ the *k^{th} copy of H_2* in G.

Now, given any vertex-coloring of G with colors 1 and 2, we consider two cases. If any copy of H_2 has all of its vertices colored 2, then we are done (this copy is an induced H_2 in G with all its vertices colored 2). Otherwise, *every* copy of H_2 in G has at least one vertex colored 1. For each $i \in [n_1]$, choose a single vertex v_i colored 1 from the i^{th} copy of H_2 in G. The graph $G[\{v_i \mid i \in [n_1]\}]$ is an induced H_1 in G with all its vertices colored 1. Indeed, our construction gives an edge between any vertex in the i^{th} copy of H_2 and any vertex in the j^{th} copy of H_2 if and only if i is adjacent to j in H_1. In particular, v_i is adjacent to v_j if and only if i is adjacent to j in H_1, as desired. □