Problem Set 2

MATH 777, Spring 2010, Mohr

March 25, 2010

1 Problem 1

Definition 1.1. *A graph G is* <u>*H-saturated*</u> *if, for all $xy \in \overline{G}$, $G+xy$ contains a copy of H, but G does not contain a copy of H.*

Proposition 1.2. *The star $K_{1,n}$ is the sparsest K_3-saturated graph on $n+1$ vertices.*

Proof. Observe first that no disconnected graph can be K_3-saturated, as adding an edge between two components of any graph will not introduce a K_3.

Consider now the graph $K_{1,n}$. As it is a tree, it is a sparsest connected graph on n vertices. To see that it is K_3-saturated, let e be an edge not present in G. It follows that e connectes two vertices of the n-block. Since each of these vertices is connected to the single vertex of the 1-block, we see that the introduction of e forms a (unique) K_3.

In fact, $K_{1,n}$ is the *unique* sparsest graph on $n + 1$ vertices. Indeed, any other tree on $n + 1$ vertices must possess a path of length 3, and the introduction of an edge between the endpoints of this path will not form a K_3. □

2 Problem 2

Proposition 2.1. *The number of K_3-free graphs on n vertices is $2^{n^2(1/4+o(1))}$.*

Proof. Let G be a graph on n vertices having an ϵ-regular patition $V_1, \dots, V_k$ where each block is of size ℓ. For each $1 \le i \le k$, $||V_i|| \le \binom{l}{2}$. Hence, the total number of edges within blocks is bounded by

$$\sum_{i=1}^{k} ||V_i|| \le k\binom{\ell}{2}$$

$$
\begin{aligned}
&\leq \frac{n}{\ell}\binom{\ell}{2} \\
&= \frac{n}{\ell}\frac{\ell(\ell-1)}{2} \\
&= \frac{n(\ell-1)}{2} \\
&= o(n^2).
\end{aligned}
$$

The number of edges between distinct blocks V_i and V_j is at most $\ell^2||R||$, where R denotes the regularity graph of G induced by the V_i. If we wish to avoid triangles, then Turan's theorem gives $||R|| \leq \frac{k^2}{4}$. Hence,

$$
\begin{aligned}
||G|| &\leq \frac{\ell^2 k^2}{4} + o(n^2) \\
&\leq \frac{n^2}{4} + o(n^2) \\
&= n^2\left(\frac{1}{4} + o(1)\right),
\end{aligned}
$$

and so the number of triangle-free graphs is bounded by $2^{n^2\left(\frac{1}{4}+o(1)\right)}$, as desired.

□

3 Problem 3

Proposition 3.1. *Let $\Gamma = K^4 - e$. The maximum number of edges in a graph on n vertices with no Γ minor is*

$$
\begin{cases}
\frac{3n-1}{2} & \text{if } n \text{ is odd} \\
\frac{3n-4}{2} & \text{if } n \text{ is even.}
\end{cases}
$$

Proof. Let G be a graph on n vertices. Evidently, if G contains a Γ minor, then G has a pair of cycles sharing a common edge. We claim that this necessary condition is also sufficient to guarantee the presence of a Γ minor. To see this, suppose that G contains a pair of cycles C_1 and C_2 sharing a common edge e. Let H be the subgraph $C_1 \cup C_2$. By performing successive contractions at edges of H other than the common edge e, we can produce two triangles sharing the edge e. Hence, G contains a Γ minor.

By the previous observation, if G is extremal for n and Γ, then G can be obtained by pasting cycles and single edges (henceforth a cycle of length

1) along vertices. We first claim that, if $||G||$ is to be maximal, then G need not contain any cycles of length 4 or larger. Suppose, for the purpose of contradiction, that G does contain a cycle $v_1v_2\dots v_nv_1$ for $n \geq 4$. Construct a graph G' by deleting the edge v_3v_4 and introducing the edge v_1v_3. Evidently $||G|| = ||G'||$ and G' has one fewer cycle with length greater than 3. Furthermore, G' maintains the property of being constructible by pasting cycles along vertices. Proceeding in this way, we can remove all cycles of length greater than 3 without reducing the total number of edges in G.

We show next that, if G is extremal for n and Γ, then it should possess at most one cycle of length 1. As before, assume for the purpose of contradiction that G possesses two or more cycles of length 1. Denote two of these cycles by u_1u_2 and v_1v_2. Since G has no pair of cycles sharing a common edge, it follows that G contracted at, say, u_1u_2 also has this property. The graph resulting from this contraction is only on $n-1$ vertices, and so we construct a new graph G' on n vertices by adding a new vertex w along with the edges v_1w and v_2w. We see that G' has two fewer cycles of length 1 but has one more edge than G.

Taken together, we see that G is obtained by pasting together cycles of length 3 and possibly one cycle of length 1 along vertices. Now, if n is odd, G must have no cycle of length 1. Hence, we construct G by beginning with a cycle of length 3, and then appending a cycle of length 3 for each additional two vertices. This construction yields

$$\begin{aligned} ||G|| &= 3 + 3\frac{n-3}{2} \\ &= \frac{3n-3}{2}. \end{aligned}$$

If n is even, G will possess a single cycle of length 1. Hence, we construct G by beginning with a cycle of length 1, and then appending a cycle of length 3 for each additional two vertices. This construction yields

$$\begin{aligned} ||G|| &= 1 + 3\frac{n-2}{2} \\ &= \frac{3n-4}{2}. \end{aligned}$$

□

4 Problem 4

Proposition 4.1. *Define $k_3(G)$ to be the number of triangles in G, i.e., the number of sets $S \subset V(G)$ with $|S| = 3$ so that $G[S] \cong K^3$. For any graph G* *on n vertices,*

$$k_3(G) + k_3(\overline{G}) \geq \frac{n(n-1)(n-5)}{24}.$$

Proof. Let $k_2(G)$ denote the number of sets S with $|S| = 3$ so that $G[S] \cong P_2$. Let $k_1(G)$ denote the number of sets S with $|S| = 3$ so that $G[S] \cong P_1$. Finally, let $k_0(G)$ denote the number of sets S with $|S| = 3$ so that $G[S] \cong \overline{K^3}$. We have,

$$k_3(G) = \binom{n}{3} - k_2(G) - k_1(G) - k_0(G),$$

since $\binom{n}{3}$ gives all choices of three vertices from G and we subtract all possible configuarations on three vertices that are not K^3s. Similarly,

$$k_3(\overline{G}) = \binom{n}{3} - k_2(\overline{G}) - k_1(\overline{G}) - k_0(\overline{G}),$$

and so

$$\begin{aligned}
k_3(G) + k_3(\overline{G}) &= \binom{n}{3} - k_2(G) - k_1(G) - k_0(G) + \binom{n}{3} - k_2(\overline{G}) - k_1(\overline{G}) - k_0(\overline{G}) \\
&= \binom{n}{3} - k_2(G) - k_2(\overline{G}) - k_3(\overline{G}) + \binom{n}{3} - k_2(\overline{G}) - k_2(G) - k_3(G) \\
&= 2\binom{n}{3} - 2k_2(G) - 2k_2(\overline{G}) - k_3(\overline{G}) - k_3(G) \\
2k_3(G) + 2k_3(\overline{G}) &= 2\binom{n}{3} - 2k_2(G) - 2k_2(\overline{G}) \\
k_3(G) + k_3(\overline{G}) &= \binom{n}{3} - k_2(G) - k_2(\overline{G}).
\end{aligned}$$

Next, we compute $k_2(G)$ by visiting each vertex and considering all the ways to select any two of its neighbors. This is an overcount, however, as a K_3 could be formed for certain choices of vertices. By this method, each K_3 is counted precisely three times (once for of each of its vertices). Hence,

$$\begin{aligned}
k_2(G) &= \sum_{v \in G} \binom{\deg(v)}{2} - 3k_3(G) \\
&= \frac{1}{2} \sum_{v \in G} (\deg(v)^2 - \deg(v)) - 3k_3(G) \\
&= \frac{1}{2} \sum_{v \in G} \deg(v)^2 - m - 3k_3(G),
\end{aligned}$$

where we let m denote the number of edges in G. In the complement, we select two vertices that are *not* neighbors of the current vertex. Hence,

$$\begin{aligned}
k_2(\overline{G}) &= \sum_{v\in G}\binom{n-1-\deg(v)}{2} - 3k_3(\overline{G}) \\
&= \frac{1}{2}\sum_{v\in G}(n-1-\deg(v))(n-2-\deg(v)) - 3k_3(\overline{G}) \\
&= \frac{1}{2}\sum_{v\in G}(n^2-3n-2n\deg(v)+2+3\deg(v)+\deg(v)^2) - 3k_3(\overline{G}) \\
&= \frac{1}{2}\sum_{v\in G}(n^2-3n+2) + \frac{1}{2}\sum_{v\in G}(\deg(v)^2+(3-2n)\deg(v)) - 3k_3(\overline{G}) \\
&= \frac{1}{2}n(n-1)(n-2) + \frac{1}{2}\sum_{v\in G}\deg(v)^2 + (3-2n)m - 3k_3(\overline{G}) \\
&= 3\binom{n}{3} + \frac{1}{2}\sum_{v\in G}\deg(v)^2 + (3-2n)m - 3k_3(\overline{G}).
\end{aligned}$$

Now,

$$\begin{aligned}
k_3(G)+k_3(\overline{G}) &= \binom{n}{3} - k_2(G) - k_2(\overline{G}) \\
&= \binom{n}{3} - \left[\frac{1}{2}\sum_{v\in G}\deg(v)^2 - m - 3k_3(G)\right] \\
&\quad - \left[3\binom{n}{3} + \frac{1}{2}\sum_{v\in G}\deg(v)^2 + (3-2n)m - 3k_3(\overline{G})\right] \\
&= -2\binom{n}{3} - \sum_{v\in G}\deg(v)^2 + 2(n-1)m + 3k_3(G) + 3k_3(\overline{G}) \\
-2k_3(G) - 2k_3(\overline{G}) &= -2\binom{n}{3} - \sum_{v\in G}\deg(v)^2 + 2(n-1)m \\
k_3(G)+k_3(\overline{G}) &= \binom{n}{3} + \frac{1}{2}\sum_{v\in G}\deg(v)^2 - (n-1)m.
\end{aligned}$$

Now, by a corollary of the Cauchy-Schwarz inequality,

$$\sum_{v\in G}\deg(v)^2 \geq \frac{\left(\sum_{v\in G}\deg(v)\right)^2}{\sum_{v\in G}1}$$

$$= \frac{4m^2}{n}.$$

Hence,

$$\begin{aligned} k_3(G) + k_3(\overline{G}) &= \binom{n}{3} + \frac{1}{2}\sum_{v \in G} \deg(v)^2 - (n-1)m \\ &\geq \binom{n}{3} + \frac{2m^2}{n} - (n-1)m. \end{aligned}$$

Now, $\frac{2m^2}{n} - (n-1)m$ is concave up, so we minimize it by taking the derivative with respect to m.

$$\begin{aligned} \frac{4m}{n} - (n-1) = 0 \\ m = \frac{n(n-1)}{4}. \end{aligned}$$

Using this value for m gives,

$$\begin{aligned} k_3(G) + k_3(\overline{G}) &\geq \binom{n}{3} + \frac{2m^2}{n} - (n-1)m \\ &\geq \binom{n}{3} + \frac{1}{8}n(n-1)^2 - \frac{1}{4}n(n-1)^2 \\ &= \frac{1}{6}n(n-1)(n-2) - \frac{1}{8}n(n-1)^2 \\ &= \frac{1}{24}n(n-1)(n-5), \end{aligned}$$

as desired. □

5 Problem 5

Proposition 5.1. *If H is the graph $(\{a,b,c,d\}, \{ab, bc, ac, ad\})$ (the "paw"), then*

$$\mathrm{ex}(n, H) = \begin{cases} \binom{n}{2} & \text{if } n \leq 3 \\ t_2(n) & \text{if } n \geq 4. \end{cases}$$

Proof. Let G be a paw-free graph on n vertices. For $n \leq 3$, G cannot possibly contain a paw, and so we take G to be the complete graph (which has $\binom{n}{2}$ edges).

Suppose now that $n \geq 4$. We first note that if G contains a triangle, this triangle must be isolated. Indeed, if any vertex of a triangle is incident to any other vertex in G, then G contains the paw as a subgraph. Hence, G is either triangle-free or contains some number of isolated triangles.

If G is triangle-free, then Turan's theorem guarantees that $\mathrm{ex}(n, H) = \mathrm{ex}(n, K^3) = t_2(n)$. We show that this is indeed optimal.

Suppose now that G contains at least one isolated triangle. If G contains no triangle-free component, then the previous remarks imply that G is a collection of isolated triangles. In this case, G has only n edges, which is less than $t_2(n)$. If G does contain a triangle-free component, we can construct a new graph having more edges by removing an edge xy from the triangle and adding the edges xv and yv for some vertex v in the triangle-free component. Evidently, the new component arising from this operation is still triangle-free (and so paw-free), and G has gained an edge. Proceeding in this way, we remove all isolated triangles from G while increasing the number of edges in G at each step. Hence, we conclude that if G is to be extremal for n and H, it must be triangle-free, and so $\mathrm{ex}(n, H) = t_2(n)$. □

6 Problem 6

Proposition 6.1. *For all* $r, n \in \mathbb{N}$, $\mathrm{ex}(n, K_{1,r}) = \lfloor \frac{1}{2} n(r-1) \rfloor$.

Proof. Evidently, $\mathrm{ex}(n, K_{1,r}) \leq \lfloor \frac{1}{2} n(r-1) \rfloor$, as this is the number is reached by forming an $(r-1)$-regular graph on n vertices, if possible. We show that this bound is attainable.

Let the vertices of G be labelled $1, \ldots, n$. We consider first the case where r is odd. Suppose that $r = 3$. For each vertex i in G, include the edge between i and $i+1$. This gives a 2-regular graph on n vertices avoiding a $K_{1,3}$ and has $n = \frac{1}{2} n(3-1)$ edges, as desired. Next, suppose $r = 5$. Beginning with the previously constructed graph, add the edge between i and $i+2$ for each i in G. As there is no duplication of edges, we indeed add an additional n edges in this manner, and so G is 4-regular on n vertices avoiding a $K_{1,5}$. Hence, G has $2n = \frac{1}{2} n(5-1)$ edges, as desired. Proceed in this way, at the j^{th} stage adding the edge between i and $i+j$ for each i in G. As there is no duplication of edges in this construction, we indeed add n edges at every stage. Hence, stage j produces a $2j$-regular graph avoiding a $K_{1,2j+1}$ and having $jn = \frac{1}{2} n(2j+1-1)$ edges, as desired.

We now consider the case where r is even. Suppose that $r = 2$. For each odd vertex i, add the edge between i and $i+1$. This gives a maximum matching of G, and thus avoids a $K_{1,2}$ and has size $\lfloor \frac{1}{2} n \rfloor = \lfloor \frac{1}{2} n(2-1) \rfloor$, as desired. For larger even $r = 2k$, perform the procedure as described for odd r for the first $k-1$ stages. This gives a graph having $n(k-1)$ edges. Finally, add in the matching as described in the $r = 2$ case. This gives an additional $\lfloor \frac{1}{2}n \rfloor$ edges, for a total of $\lfloor \frac{1}{2}n(r-1) \rfloor$, as desired. □

7 Problem 7

Definition 7.1. *The* <u>*upper density*</u> *of an infinite graph G is the infimum of all reals α such that the finite graphs $H \subseteq G$ with $||H||\binom{|H|}{2}^{-1} > \alpha$ have bounded order.*

Proposition 7.2. *The upper density always takes on of the countably-many values 0, 1, $\frac{1}{2}$, $\frac{2}{3}$, $\frac{3}{4}$, ...*

Proof. Suppose, for the purpose of contradiction, that the upper density α does not take on one of the above values. Let r be maximal with $\alpha > 1 - \frac{1}{r-1}$. Since α is defined in terms of an infimum, there is a fixed ϵ such that, for all $m \in \mathbb{N}$, there is a graph H on at least m vertices satisfying

$$||H||\binom{|H|}{2}^{-1} > 1 - \frac{1}{r-1} + \epsilon.$$

Letting n denote $|H|$, we have

$$\begin{aligned} ||H|| &> \binom{|H|}{2}\left(1 - \frac{1}{r-1} + \epsilon\right) \\ &\approx \frac{n^2}{2}\left(1 - \frac{1}{r-1} + \epsilon\right) \\ &= \frac{r-2}{r-1} \cdot \frac{n^2}{2} + \epsilon\frac{n^2}{2} \\ &\geq t_{r-1}(n) + \epsilon' n^2. \end{aligned}$$

By the Erdős-Stone theorem, H contains K_s^r as a subgraph for all $s \geq 1$. It follows that

$$\begin{aligned} ||H||\binom{|H|}{2}^{-1} &\approx \binom{r}{2}s^2\binom{rs}{2}^{-1} \\ &= \frac{r(r-1)s^2}{rs(rs-1)} \\ &= \frac{(r-1)s}{rs-1} \end{aligned}$$

$$= \frac{r-1}{r-\frac{1}{s}},$$

which tends to $1-\frac{1}{r}$. As r was chosen to be maximal with $\alpha > 1-\frac{1}{r-1}$, we have that $\alpha = 1-\frac{1}{r}$, and so α takes on one of the desired values. □

8 Problem 8

Proposition 8.1. *For any tree T, $\mathrm{ex}(n,T) \leq n(|T|-3)$.*

Proof. Let T be any tree and let G be extremal for n and T. We know that if $\delta(G) \geq |T|-1$, then G contains a copy of T. Moreover, G contains a subgraph H satisfying $\delta(H) > \epsilon(H) \geq \epsilon(G)$. Taken together, we see that if $\epsilon(G) \geq |T|-2$, then G has a subgraph H with $\delta(H) \geq |T|-1$ (and so $T \subseteq H \subseteq G$). Thus, we require that

$$\begin{aligned} &\epsilon(G) < |T|-2 \\ &\frac{||G||}{n} < |T|-2 \\ &||G|| < n(|T|-2). \end{aligned}$$

Therefore, $\mathrm{ex}(n,T) \leq n(|T|-3)$. □

9 Problem 9

Theorem 9.1. *(Mader '67) If G has average degree at least 2^{r-2}, then G contains a K^r minor.*

Proof. (Incomplete) Let H be a minor of G that is minimal with respect to $\epsilon(H) \geq \epsilon(G)$. Let $e = uv$ be an edge in H. It follows that

$$\begin{aligned} \epsilon(G) &> \epsilon(H \setminus e) && \text{(by minimality of } H) \\ &= \frac{||H \setminus e||}{|H \setminus e|} \\ &\geq \frac{||H|| - 1 - |N(u) \cap N(v)|}{|H|-1}, \end{aligned}$$

and so

$$\begin{aligned} |N(u) \cap N(v)| &> ||H|| - (|H|-1)\epsilon(G) - 1 \\ &= ||H|| - \epsilon(G)|H| + \epsilon(G) - 1 \end{aligned}$$

$$
\begin{aligned}
&\geq ||H|| - \epsilon(H)|H| + \epsilon(G) - 1 \qquad \text{(since } \epsilon(H) \geq \epsilon(G)) \\
&= ||H|| - \frac{||H||}{|H|}|H| + \epsilon(G) - 1 \\
&= \epsilon(G) - 1 \\
&\geq 2^{r-2} - 1.
\end{aligned}
$$

Since the above inequality is strict, we have that $|N(u) \cap N(v)| \geq 2^{r-2}$. □

10 Problem 10

Proposition 10.1. *If G is a graph with $\delta(G) \geq 3$, then G contains a TK^4.*

Proof. We proceed by establishing the contrapositive. To that end, suppose that G does not contain a TK^4. Denote by G' a graph containing G that is edge-maximal with respect to avoiding a TK^4. We know that G' can be constructed recursively from triangles by pasting along single edges. Now, as the pasting takes place along single edges, the most recently pasted triangle at any stage of the construction has a vertex of degree 2 (namely, that vertex not belonging to the edge along which we are currently pasting). Hence, $\delta(G) \leq \delta(G') = 2$, thus establishing the contrapositive. □