Problem Set 1

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1 Problem 1

Definition 1.1. *We say that a graph G has an S^1-flow if it has a $\mathbb{C}$-circulation f so that $|f(\overrightarrow{e})| = 1$ for all $\overrightarrow{e} \in \overrightarrow{E}$.*

Lemma 1.2. *Let $z_1 \in \mathbb{C}$ with $|z| = 1$. There are unique $z_2, z_3 \in \mathbb{C}$ with $|z_2| = |z_3| = 1$ satisfying $z_1 + z_2 + z_3 = 0$.*

Proof. Observe first that, if z_1, z_2, and z_3 satisfy the desired equations, then the complex numbers resulting from the application of any single rotation to all of z_1, z_2 and z_3 also satisfy the equations. Hence, we assume, without loss of generality, that $z_1 = 1$. Let $z_2 = a + bi$ and $z_3 = c + di$. To ensure $|z_2| = |z_3| = 1$, we require $\sqrt{a^2+b^2} = \sqrt{c^2+d^2} = 1$. More simply,

$$a^2 + b^2 = c^2 + d^2 = 1 \tag{1}$$

To ensure $1 + z_2 + z_3 = 0$, we require $(a + bi) + (c + di) = -1$. In other words,

$$a + c = -1 \tag{2}$$
$$b + d = 0. \tag{3}$$

It follows that

$$\begin{aligned} a^2 + b^2 &= c^2 + d^2 && \text{(by (1))} \\ a^2 &= c^2 && \text{(by (3))} \\ a &= \pm c. \end{aligned}$$

Combining this last line with (2), we see that $a = c = -\frac{1}{2}$. With this new information, we have

$$\begin{aligned} \left(\frac{1}{2}\right)^2 + b^2 &= 1 && \text{(by (1))} \\ b^2 &= \frac{3}{4} \\ b &= \pm\frac{\sqrt{3}}{2}. \end{aligned}$$

Similarly, $d = \pm\frac{\sqrt{3}}{2}$. By (3), we take, without loss of generality, $b = \frac{\sqrt{3}}{2}$ and $d = -\frac{\sqrt{3}}{2}$. Hence, $z_2 = \frac{-1+i\sqrt{3}}{2}$ and $z_3 = \frac{-1-i\sqrt{3}}{2}$ provide the unique solution when $z_1 = 1$. By the remarks at the beginning of the proof, we are guaranteed a unique solution for any choice of z_1 satisfying $|z_1| = 1$. □

Proposition 1.3. *If G is cubic, it has an S^1-flow if and only if it has a 3-flow.*

Proof. (⇒) Let G be cubic and possess an S_1-flow f. We show that G is bipartite, and so conclude that G possesses a 3-flow.

First, let $v \in V(G)$ and let z_1, z_2, and z_3 be the complex-valued flow leaving v along each of its incident edges. Evidently, $|z_1| = |z_2| = |z_3| = 1$ and $z_1+z_2+z_3 = 0$. Now, consider the neighbor w of v such that the edge from v to w has flow, say, z_1. By the lemma, there is a unique assignment of flow for the other two edges incident to w satisfying the desired properties for an S_1-flow, namely $-z_2$ and $-z_3$. Continuing in this fashion, we see that f only takes on the values $\pm z_1$, $\pm z_2$, and $\pm z_3$.

Let A denote the set of vertices of G that have flow of z_1 directed outward and B denote the set of vertices of G that have flow of z_1 directed inward (i.e. flow of $-z_1$ directed outward). By the lemma, every vertex must belong to one precisely one of these sets. Hence, A and B partition $V(G)$. Now, there can be no edge between two vertices of A, since the flow of z_1 along this edge must be leaving one of the vertices and entering the other (hence, one of them belongs in B). Similarly, there are no edges between vertices of B. Therefore, G is bipartite, and so possesses a 3-flow.

(⇐) Let G be cubic and possess a 3-flow. We construct an S_1-flow f on G.

First, observe that G is bipartite, as it is cubic and possesses a 3-flow. Let A and B denote the partite sets of $V(G)$. For any edge $e \in E(G)$, let $\overrightarrow{e}$ denote the direction of e from A to B.

Now, since G is cubic, it possesses a 1-factor M_0. For every edge $e \in M_0$, define $f(\overrightarrow{e}) = 1$ and $f(\overleftarrow{e}) = -1$. Note that $|1| = |-1| = 1$.

Next, consider $G \setminus M_0$. This graph is 2-regular, and so possesses a 1-factor M_1. For every edge $e \in M_1$, define $f(\overrightarrow{e}) = \omega$ and $f(\overleftarrow{e}) = -\omega$, where ω is a primitive cube root of unity. Note that $|\omega| = |-\omega| = 1$.

Finally, consider $G \setminus (M_0 \cup M_1)$. This graph is 1-regular, and so possesses a 1-factor M_2. For every edge $e \in M_2$, define $f(\overrightarrow{e}) = \omega^2$ and $f(\overleftarrow{e}) = -\omega^2$. Note that $|\omega^2| = |-\omega^2| = 1$.

Since $M_0 \cup M_1 \cup M_2 = E(G)$, f is now defined for every edge in G. Evidently, $f(\overrightarrow{e}) = -f(\overleftarrow{e})$ for all $e \in E(G)$. We see also that every vertex in $v \in A$ has $f(v,V) = 1+\omega+\omega^2 = 0$. Similarly, for all $v \in B$, we have $f(v,V) = -1-\omega-\omega^2 = 0$. Therefore, f is an S_1-flow on G. □

2 Problem 2

Lemma 2.1. *Every connected even graph has a covering by edge-disjoint cycles.*

Proof. Let G be an even graph on n vertices. We proceed by induction on n.

If $n = 3$ (the smallest possible value for an even simple graph), then G is itself a cycle.

Suppose now that the claim holds for $n = k$ and let G be even on $k+1$ vertices. It must be that either G is itself a cycle (in which case there is nothing to show) or G contains a cycle C having fewer than $k+1$ vertices (if G contains no such cycle, then it cannot possibly have an Euler tour). Observe that $G \setminus C$ is an even graph, since we reduce the degree of every vertex in the even graph G by 0 or 2. By the inductive hypothesis, $G \setminus C$ can be covered by edge-disjoint cycles. Taking these cycles together with C gives the desired covering of G. □

Lemma 2.2. *Let G be the union of two even graphs G_1 and G_2. The graph $(V(G), E(G_1)\Delta E(G_2))$ is even.*

Proof. As each of G_1 and G_2 are even, they each possess a 2-flow. For reasons to become apparent later, define the flow f_1 on each edge of $e \in G_1$ to be $f_1(e) = (0,1) \in \mathbb{Z}_2 \times \mathbb{Z}_2$. Similarly, $f_2(e) = (1,0) \in \mathbb{Z}_2 \times \mathbb{Z}_2$ for each edge $e \in G_2$. The function $f = f_1 + f_2$ is a 4-flow on G, where we assume the f_i are defined to be 0 on edges outside their respective domains. To see this, observe first that, for all $e \in G$, $f(\overrightarrow{e}) = -f(\overleftarrow{e})$ since every element of $Z_2 \times Z_2$ is self-inverse. Now, let $V = V(G)$. For all vertices $v \in V$,

$$\begin{aligned} f(v,V) &= f_1(v,V) + f_2(v,V) \\ &= \sum_{u \in N(v)} f_1(uv) + \sum_{u \in N(v)} f_2(uv) \\ &= \sum_{\substack{u \in N(v) \\ uv \in E(G_1)}} f_1(uv) + \sum_{\substack{u \in N(v) \\ uv \notin E(G_1)}} f_1(uv) + \sum_{\substack{u \in N(v) \\ uv \in E(G_2)}} f_2(uv) + \sum_{\substack{u \in N(v) \\ uv \notin E(G_2)}} f_2(uv) \\ &= 0 + 0 + 0 + 0 \\ &= 0 \end{aligned}$$

By this construction, the edges having flow $(1,1)$ are precisely the edges in $E(G_1) \cap E(G_2)$. Moreover, for every vertex $v \in G$, the total number of edges incident to v having flow other than $(1,1)$ must be even (else the sum of the flows at v cannot possibly be 0). Hence, removing the edges with flow $(1,1)$ (i.e. removing $E(G_1) \cap E(G_2)$) leaves an even graph, as desired. □

Proposition 2.3. *If G has a 4-flow, then it contains a collection of cycles $\{C_1, \ldots, C_N\}$ covering all of the edges (i.e., $\bigcup_{j=1}^{N} E(C_j) = E(G)$) so that*

$$\sum_{j=1}^{N} \|C_j\| \le \frac{4}{3}\|G\|.$$

Proof. Let G possess a 4-flow, which we will view as a $Z_2 \times Z_2$ flow. Observe first that G is the union of two even graphs G_1 and G_2. By 2.1, there are sets of edge-disjoint cycles $\mathcal{C}_1$ and $\mathcal{C}_2$ covering G_1 and G_2, respectively. Now, using the flow constructed in 2.2, let A be the set of edges having flow $(0,1)$, B be the set of edges having flow $(1,0)$, and C be the set of edges having flow $(1,1)$. Evidently, $||G|| = |A| + |B| + |C|$. Moreover, we have that $C = E(G_1) \cap E(G_2)$.

Now, it must be that one of $|A|$, $|B|$, or $|C|$ is at most $\frac{1}{3}||G||$. If $|C| \le \frac{1}{3}||G||$, then $\mathcal{C}_1 \cup \mathcal{C}_2$ is the desired covering, as

$$\begin{aligned}\sum_{C \in \mathcal{C}_1 \cup \mathcal{C}_2} ||C|| &= ||G_1|| + ||G_2|| \\ &= (|A| + |C|) + (|B| + |C|) \\ &= (|A| + |B| + |C|) + |C| \\ &\le ||G|| + \frac{1}{3}||G|| \\ &= \frac{4}{3}||G||.\end{aligned}$$

We conclude the proof by demonstrating that one can always construct a 4-flow in which the number of edges of G having flow $(1,1)$ is at most $\frac{1}{3}||G||$. To that end, let A, B, and C be as above. The case where $|C| \le \frac{1}{3}||G||$ was handled previously, and so we assume, without loss of generality, that $|A| \le \frac{1}{3}||G||$. Now, let $G' = (V(G), A \cup B)$ (i.e. $G' = (V(G), E(G_1)\Delta E(G_2))$). By 2.2, G' is even, and so possesses a 2-flow f_0. Define $f_0(e) = (0,1)$ for all $e \in G'$. As G_1 is even, define the 2-flow f_1 where $f_1(e) = (1,0)$ for all $e \in G_1$. As in the proof of 2.2, extend these to a 4-flow f on G by letting $f = f_0 + f_1$. Evidently, the set of edges having flow $(1,1)$ under f is precisely A. Since we assumed $|A| \le \frac{1}{3}||G||$, the claim follows as before. □

3 Problem 3

Definition 3.1. *For $x \in \mathbb{R}$, a function $f : \overrightarrow{E} \to \mathbb{R}$ is an* <u>*x-flow*</u> *if it satisfies*

- $f(\overrightarrow{e}) = -f(\overleftarrow{e})$ *for all $e \in E(G)$,*
- $f(v, V(G)) = 0$ *for all $v \in V(G)$, and*
- $1 \le |f(\overrightarrow{e})| \le x - 1$ *for all $\overrightarrow{e} \in \overrightarrow{E}$.*

Proposition 3.2. *If a graph has an x-flow, then it has an (integral) $\lceil x \rceil$-flow.*

Proof. We proceed by induction on the number of edges of G having non-integral flow.

If there are no edges having non-integral flow, then the claim holds trivially.

Suppose now that the claim holds for graphs having k edges with non-integral flow. Let f be an x-flow on G having $k+1$ non-integral edges. Consider the set $\mathcal{C}$ of all cycles in G in which every edge is non-integral. Observe that such a cycle must exist, since if a vertex v_1 sends non-integral flow to v_2, v_2 must send non-integral flow to some v_3 by Kirchoff's Law. Hence, non-integral flow must continue to be routed through G until it eventually returns to v_1, thus forming a cycle.

Now, let e_1 be an edge minimizing $f(\overrightarrow{e}) - \lfloor f(\overrightarrow{e}) \rfloor$ taken over all edges e belonging to cycles of $\mathcal{C}$, where we assume $\overrightarrow{e}$ is the direction of e having positive flow. Let y_1 denote this minimum value. Similarly, let e_2 be an edge minimizing $\lceil f(\overrightarrow{e}) \rceil - f(\overrightarrow{e})$ and denote this minimum value by y_2. Finally, let

$$e_0 = \begin{cases} e_1 & \text{if } y_1 \leq y_2 \\ e_2 & \text{otherwise} \end{cases}$$

and let $y_0 = \min\{y_1, y_2\}$. Loosely speaking, e_0 is the edge beloning to a cycle of $\mathcal{C}$ whose flow is closest to an integer and y_0 is the distance this flow is from being an integer. Now, let $\overrightarrow{e_0}$ belong to the directed cycle $\overrightarrow{C} = \overrightarrow{e_0} \cdots \overrightarrow{e_l}$. Note that $f(\overrightarrow{e_0})$ is positive by assumption, but this may not be the case for the remaining $\overrightarrow{e_i}$. For all $e \in C$, let $\overrightarrow{e}$ denote the direction specified in $\overrightarrow{C}$. For $e \in G \setminus C$, fix a direction $\overrightarrow{e}$ arbitrarily. We construct a new flow f' where, for all $e \in G$,

$$f'(\overrightarrow{e}) = \begin{cases} f(\overrightarrow{e}) & \text{if } e \notin C \\ f(\overrightarrow{e}) - y_0 & \text{if } e \in C \text{ and } e_0 = e_1 \\ f(\overrightarrow{e}) + y_0 & \text{if } e \in C \text{ and } e_0 = e_2 \end{cases}$$

and we take the appropriate value for $f'(\overleftarrow{e})$ so as to satisfy $f'(\overrightarrow{e}) = -f'(\overleftarrow{e})$. By our choice of y_0, we have that $1 \leq |f'(\overrightarrow{e})| \leq x - 1$ for all $e \in G$. Furthermore, $f'(v, V) = f(v, V) = 0$ for all $v \in V$, since we add (or subtract) a constant value from the flow at every edge of a cycle.

Now, $f'(\overrightarrow{e_0})$ is integral by our choice of y_0, and so f' is an x-flow on G having only k non-integral edges. By the inductive hypothesis, we conclude that G possesses a $\lceil x \rceil$-flow. □

4 Problem 4

Lemma 4.1. *If two graphs G and H both admit a k-flow, then $G \square H$ admits a k-flow.*

Proof. Let g and h be k-flows on G and H, respectively. Call $e \in E(G \square H)$ a G-edge if its endpoints are of the form (x, y) and (x', y) for some $x, x' \in G$ and

$y \in H$. Similarly, call e an H-edge if its endpoints are of the form (x, y) and (x, y') for some $x \in G$ and $y, y' \in H$.

Define the function f by letting

$$f(\overrightarrow{e}) = \begin{cases} g(\overrightarrow{e}) & \text{if } e \text{ is a } G\text{-edge} \\ h(\overrightarrow{e}) & \text{if } e \text{ is an } H\text{-edge.} \end{cases}$$

As both g and h are k-flows, we have immediately that f is nowhere zero and, for all $\overrightarrow{e} \in \overrightarrow{E(G \square H)}$, $0 < |f(\overrightarrow{e})| < k$ and $f(\overrightarrow{e}) = -f(\overleftarrow{e})$. It remains to show that, for all $v \in V(G \square H)$, $f(v, V(G \square H)) = 0$. To that end, let $v = (x, y) \in V(G \square H)$. Observe that there is a one-to-one correspondence between the G-edges leaving v and the edges in $E(G)$ leaving x. A similar correspondence exists between the H-edges leaving v and the edges in $E(H)$ leaving y. It follows that

$$\begin{aligned} f(v, V(G \square H)) &= g(x, V(G)) + h(y, V(H)) \\ &= 0 + 0 \\ &= 0, \end{aligned}$$

as desired. Therefore, f is a k-flow on $G \square H$. □

Lemma 4.2. *For $n \geq 1$, Q_n is bipartite.*

Proof. Let

$$\begin{aligned} A &= \{v \in V(Q_n) \mid v \text{ has an even number of 1 bits}\} \\ B &= \{v \in V(Q_n) \mid v \text{ has an odd number of 1 bits}\}. \end{aligned}$$

Evidently, A and B partition $V(Q_n)$. No two vertices of A are adjacent, as any pair must differ in at least two bits. For the same reason, no two vertices of B are adjacent. Therefore, Q_n is bipartite for all $n \geq 1$. □

Proposition 4.3. *For all $n \geq 1$,*

$$\varphi(Q_n) = \begin{cases} \infty & \text{if } n = 1 \\ 2 & \text{if } n \geq 2 \text{ even} \\ 3 & \text{if } n \geq 3 \text{ odd.} \end{cases}$$

Proof. The graph Q_1 is a single edge, and so contains a bridge. Hence, there can be no flow on this graph. In this case, we say that Q_1 has infinite flow number.

For even values of n, Q_n is an even graph (as it is n-regular), and so possesses a 2-flow.

Next, observe that Q_3 is 3-regular and bipartite, and so possesses a 3-flow. For larger odd values n, note that

$$\begin{aligned} Q_n &= Q_{2k+1} && \text{(for some } k\text{)} \\ &= (Q_{2k-2} \square Q_1) \square Q_1) \square Q_1 \end{aligned}$$

$$= Q_{2k-2} \square (Q_1 \square Q_1 \square Q_1) \qquad \text{(by associativity of } \square)$$
$$= Q_{2k-2} \square Q_3.$$

Now, Q_{2k-2} is even, and so possesses a 2-flow (hence, a 3-flow). As previously observed, Q_3 also possesses a 3-flow. By the lemma, Q_n possesses a 3-flow, since it is the Cartesian product of two graphs possessing a 3-flow. Furthermore, Q_n cannot possess a 2-flow, as it is not an even graph. Hence, $\varphi(Q_n) = 3$ for odd $n \geq 3$. □

5 Problem 5

Proposition 5.1. *Let H be a finite abelian group, G a graph, and T a spanning tree of G. Every mapping f from the directions of $E(G) \setminus E(T)$ to H that satisfies $f(\overrightarrow{e}) = -f(\overleftarrow{e})$ for all $e \in E(G) \setminus E(T)$ extends uniquely to an H-circulation on G.*

Proof. Let f be given as above and let T be rooted at some vertex r. Consider a leaf v_0 (with corresponding pendant edge $v_0 w_0$) at the lowest level of T. Let $V = V(G)$ and let $f(u, v)$ denote the flow from a vertex u to a vertex v. We have

$$\begin{aligned} f(v_0, V) &= \sum_{v \in N(v_0)} f(v_0, v) \\ &= \sum_{\substack{v \in N(v_0) \\ v \neq w_0}} f(v_0, v) + f(v_0, w_0) \qquad \text{(since } H \text{ is abelian).} \end{aligned}$$

Setting $f(v_0, w_0) = -\sum_{\substack{v \in N(v_0) \\ v \neq w_0}} f(v_0, v)$ gives $f(v_0, V) = 0$. To satisfy $f(v_0, w_0) = -f(w_0, v_0)$, we must of course set $f(w_0, v_0) = \sum_{\substack{v \in N(v_0) \\ v \neq w_0}} f(v_0, v)$.

Carry out this process levelwise (that is, assign flow to all edges of a given level, then proceed to the next level). More precisely, for each $v_k, w_k \in V$ with v_k below w_k in T, set $f(v_k, w_k) = -\sum_{\substack{v \in N(v_k) \\ v \neq w_k}} f(v_k, v)$ (since we assign flow levelwise, all the values in the summation will indeed by defined). Set also $f(w_k, v_k) = \sum_{\substack{v \in N(v_k) \\ v \neq w_k}} f(v_k, v)$. From this perspective, it is clear that the extension is unique and satisfies $f(v, w) = -f(w, v)$ for all $v, w \in V$, as there is a unique choice of flow at each $v \in V$ satisfying $f(v, V) = 0$. It is not evident, however, that $f(r, V) = 0$, as r is not below any vertex in T, and so no flow is chosen specifically to satisfy $f(r, V) = 0$. To verify that this constraint holds, observe that

$$\begin{aligned} 0 &= f(V, V) && \text{(since } f(\overrightarrow{e}) = -f(\overleftarrow{e}) \text{ for all } e \in G) \\ &= f(r, V) + f(V \setminus r, V) \\ &= f(r, V) && \text{(since } f(u, V) = 0 \text{ for all } u \neq r), \end{aligned}$$

and so f is indeed an H-circulation on G. □

6 Problem 6

Proposition 6.1. *If G is a graph with m spanning trees such that no edge of G lies in all of these trees, then $\varphi(G) \leq 2^m$.*

Proof. Let $T_1, \dots, T_m$ denote the spanning trees. For $1 \leq i \leq m$, we construct a Z_2-circulation f_i on G. Begin with 0 flow on every edge of G. For $e \in G \setminus T_i$, let C_e denote the unique cycle in $T_i \cup e$. For each $e \in G \setminus T_i$, set $f_{i,e}$ to be 1 on each edge of C_e. Now, let

$$f_i = \sum_{e \in G \setminus T_i} f_{i,e} \text{ (modulo 2).}$$

Evidently, f_i is a $\mathbb{Z}_2$-circulation on G, since it is the sum of the $\mathbb{Z}_2$-circulations $f_{i,e}$.

For each $e \in G$, define

$$f(e) =< f_1(e), \dots, f_m(e) >$$

We claim that f is a Z_2^m-flow on G. As before, $f(\overrightarrow{e}) = -f(\overleftarrow{e})$, since every element of Z_2^m is self-inverse. Now, for all $v \in G$,

$$\begin{aligned} f(v, V) &= \sum_{w \in N(v)} f(v, w) \\ &= \sum_{w \in N(v)} < f_1(v, w), \dots, f_m(v, w) > \\ &= \left\langle \sum_{w \in N(v)} f_1(v, w), \dots, \sum_{w \in N(v)} f_m(v, w) \right\rangle \\ &=< 0, \dots, 0 >, \end{aligned}$$

and so f is a Z_2^m-circulation on G. Moreover, f is actually a flow, since $f(e)$ is nonzero for all $e \in G$. To see this, observe that, for all $1 \leq i \leq m$, $f_i(e) = 1$ whenever $e \notin T_i$, since we consider each $e \in G \setminus T_i$ only once (and so increment its flow only once). Since no edge is in all of the T_i, it follows that, for every $e \in G$, $f(e)$ is nonzero in at least one coordinate. Therefore, f is indeed a Z_2^m-flow on G, and so $\varphi(G) \leq 2^m$.

□

7 Problem 7

Proposition 7.1. *Every graph with a Hamiltonian cycle has a 4-flow.*

Proof. Let G be a Hamiltonian graph with Hamilton cycle C. We construct a $Z_2 \times Z_2$ flow f on G. Note that, since every element of $Z_2 \times Z_2$ is self-inverse, we need not worry about the orientation of flow along the edges of G.

Begin by letting $f_0(e) = (0, 1)$ for all $e \in E(C)$. Evidently, f_0 is a 4-flow on C (indeed, f_0 is a 2-flow on C).

Let now $e_1 \in E(G) \setminus E(C)$ (that is, e_1 is a chord of C). Denote the endpoints of e_1 by v_1 and w_1 and let $P(e_1)$ be either of the v_1w_1-paths contained in C. For each edge $e \in E(G)$, set

$$f_1(e) = \begin{cases} f_0(e) & \text{if } e \notin P(e_1) \cup e_1 \\ (1,1) & \text{if } e = e_1 \\ (1,0) & \text{if } e \in P(e_1) \end{cases}$$

Now, for $u \notin \{v_1, w_1\}$, either $f_1(u,V) = (0,1) + (0,1)$ or $f_1(u,V) = (1,0) + (1,0)$, where $V = V(G)$. In either case, $f_1(u,V) = (0,0)$. For $u \in \{v_1, w_1\}$, $f_1(u,V) = (0,1) + (1,0) + (1,1) = (0,0)$. Thus, f_1 is a 4-flow on $C \cup e_1$.

At the k^{th} stage of this process, select a chord $e_k = v_kw_k$ and let $P(e_k)$ be one of the v_kw_k-paths contained in C. For each edge $e \in E(G)$, set

$$f_k(e) = \begin{cases} f_{k-1}(e) & \text{if } e \notin P(e_k) \cup e_k \\ (1,1) & \text{if } e = e_k \\ (0,1) & \text{if } e \in P(e_k) \text{ and } f_{k-1}(e) = (1,0) \\ (1,0) & \text{if } e \in P(e_k) \text{ and } f_{k-1}(e) = (0,1) \end{cases}$$

Note that, by our construction, only the values $(0,1)$ or $(1,0)$ may appear on C (as only $(0,1)$ appeared in f_0), and so the above defintion is valid. Now, for $u \notin P(e_k)$, $f_k(u,V) = f_{k-1}(u,V) = 0$. For $u \in P(e_k) \setminus \{v,w\}$, let l be the number of chords incident to u. If l is even, then the fact that $f_{k-1}(u) = (0,0)$ forces

$$f_{k-1}(u,V) = l(1,1) + (0,1) + (0,1),$$

where we let, without loss of generality, the edges of $P(e_k)$ that are incident to u have flow $(0,1)$ (the only important aspect is that these two edges must agree in their flow). It follows that

$$\begin{aligned} f_k(u,V) &= l(1,1) + (1,0) + (1,0) \\ &= (0,0). \end{aligned}$$

On the other hand, l is odd, then it must be that

$$f_{k-1}(u,V) = l(1,1) + (0,1) + (1,0),$$

and so

$$\begin{aligned} f_k(u,V) &= l(1,1) + (1,0) + (0,1) \\ &= (0,0). \end{aligned}$$

If $u \in \{v_k, w_k\}$, we argue similarly. Let l denote the number of chords adjacent to u excluding e_k. If l is even, then we must have, without loss of generality,

$$f_{k-1}(u,V) = l(1,1) + (0,1) + (0,1),$$

and so adding in e_k and reassigning one of the edges in C incident to u gives

$$f_k(u,V) = l(1,1) + (1,1) + (1,0) + (0,1)$$

$$= (0,0).$$

If l is odd, then

$$f_{k-1}(u,V) = l(1,1) + (0,1) + (1,0),$$

and so

$$\begin{aligned} f_k(u,V) &= l(1,1) + (1,1) + (1,0) + (0,1) \\ &= (0,0), \end{aligned}$$

where we let, without loss of generality, the edge in $P(e_k)$ have flow $(0,1)$ (the only important aspect is that one of these flows changes and the other does not). Hence, f_k is a 4-flow on $C \cup e_1 \cup \cdots \cup e_k$ for all k.

Letting r be the number of chords in G, the function f_r defines a 4-flow on G. □

8 Problem 8

Definition 8.1. *A family of (not necessarily distinct) cycles in a graph G is called a cycle double cover of G if every edge of G lies on exactly two of these cycles. The cycle double cover conjecture asserts that every bridgeless multigraph has a cycle double cover.*

Lemma 8.2. *Every connected even graph has a covering by edge-disjoint cycles.*

Proof. Let G be an even graph on n vertices. We proceed by induction on n.

If $n = 3$ (the smallest possible value for an even simple graph), then G is itself a cycle.

Suppose now that the claim holds for $n = k$ and let G be even on $k+1$ vertices. It must be that either G is itself a cycle (in which case there is nothing to show) or G contains a cycle C having fewer than $k+1$ vertices (if G contains no such cycle, then it cannot possibly have an Euler tour). Observe that $G \setminus C$ is an even graph, since we reduce the degree of every vertex in the even graph G by 0 or 2. By the inductive hypothesis, $G \setminus C$ can be covered by edge-disjoint cycles. Taking these cycles together with C gives the desired covering of G. □

Lemma 8.3. *Let G be the union of two even graphs G_1 and G_2. The graph $(V(G), E(G_1)\Delta E(G_2))$ is even.*

Proof. As each of G_1 and G_2 are even, they each possess a 2-flow. For reasons to become apparent later, define the flow f_1 on each edge of $e \in G_1$ to be $f_1(e) = (0,1)$. Similarly, $f_2(e) = (1,0)$ for each edge $e \in G_2$. The function $f = f_1 + f_2$ is a 4-flow on G, where we assume the f_i are defined to be 0 on edges outside their respective domains. To see this, observe first that, for all $e \in G$, $f(\overrightarrow{e}) = -f(\overleftarrow{e})$ since every element of $Z_2 \times Z_2$ is self-inverse. Now, let $V = V(G)$. For all vertices $v \in V$,

$$f(v,V) = f_1(v,V) + f_2(v,V)$$

$$
\begin{aligned}
&= \sum_{u \in N(v)} f_1(uv) + \sum_{u \in N(v)} f_2(uv) \\
&= \sum_{\substack{u \in N(v) \\ uv \in E(G_1)}} f_1(uv) + \sum_{\substack{u \in N(v) \\ uv \notin E(G_1)}} f_1(uv) + \sum_{\substack{u \in N(v) \\ uv \in E(G_2)}} f_2(uv) + \sum_{\substack{u \in N(v) \\ uv \notin E(G_2)}} f_2(uv) \\
&= 0 + 0 + 0 + 0 \\
&= 0
\end{aligned}
$$

By this construction, the edges having flow $(1,1)$ are precisely the edges in $E(G_1) \cap E(G_2)$. Moreover, for every vertex $v \in G$, the total number of edges incident to v having flow other than $(1,1)$ must be even (else the sum of the flows at v cannot possibly be 0). Hence, removing the edges with flow $(1,1)$ (i.e. removing $E(G_1) \cap E(G_2)$) leaves an even graph, as desired. □

Proposition 8.4. *The cycle double cover conjecture holds for graphs with a 4-flow.*

Proof. Let G be a graph possessing a 4-flow. It follows that G is the union of two even graphs G_1 and G_2, each of which possess a 2-flow. Let $G' = (V(G), E(G_1) \Delta E(G_2))$. By 8.3, G' is even, and so possesses a 2-flow. Now, define

$$
\begin{aligned}
\mathcal{C}_1 &= \text{a covering of } G_1 \text{ by edge-disjoint cycles} \\
\mathcal{C}_2 &= \text{a covering of } G_2 \text{ by edge-disjoint cycles} \\
\mathcal{C}_3 &= \text{a covering of } G' \text{ by edge-disjoint cycles,}
\end{aligned}
$$

each of which is guaranteed to exist by 8.2. It follows that $\mathcal{C} = \mathcal{C}_1 \cup \mathcal{C}_2 \cup \mathcal{C}_3$ is the desired cycle double cover, as every edge of G appears in precisely two of the $\mathcal{C}_i$. □