Final Problem Set

MATH 777, Spring 2010, Mohr

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1 Problem 1

Proposition 1.1. *The random graph $\mathcal{G}(n,p)$ is Hamiltonian asymptotically almost surely, where $p \in (0,1]$ is a constant.*

Proof. Let $G \in \mathcal{G}(n,p)$ for constant p. We establish that $\alpha(G) \leq \kappa(G)$ asymptotically almost surely, which implies that G is Hamiltonian asymptotically almost surely.

We first bound $\alpha(G)$ from above. Let $a = \frac{3}{-\ln q}\ln n + 1$. We have

$$\begin{aligned}
\mathrm{P}[\alpha(G) \geq a] &\leq \binom{n}{a} q^{\binom{a}{2}} \\
&\leq n^a q^{\frac{1}{2}a(a-1)} \\
&= \exp\left(a \ln n + \frac{1}{2}a(a-1)\ln q\right) \\
&= \exp\left(\frac{1}{2}a(2\ln n + (a-1)\ln q)\right) \\
&= \exp\left(\frac{1}{2}\left(\frac{3}{-\ln q}\ln n + 1\right)\left(2\ln n + \left(\frac{3}{-\ln q}\ln n\right)\right)\ln q\right) \\
&= \exp\left(\frac{-1}{2}\left(\frac{3}{-\ln q}\ln n + 1\right)\ln n\right) \\
&\to 0 \qquad\qquad (\text{as } n \to \infty).
\end{aligned}$$

Hence, $\alpha(G)$ is asymptotically almost surely less than $\frac{3}{-\ln q}\ln n + 1$.

We next bound $\kappa(G)$ from below. Let $k = \frac{-\ln q}{2}\frac{n}{\ln n}$. For a given subset S of $V(G)$, let X_S be the random variable indicating that S is a vertex cut of G. We have

$$\begin{aligned}
\mathrm{P}[\kappa(G) \leq k] &\leq E\left(\sum_{\substack{S \subset V(G) \\ |S|=k}} X_S\right) \\
&\leq \sum_{i=1}^{k} \binom{n}{i} q^{n-1} \\
&\leq q^{n-1} k \binom{n}{k} && \left(\text{since } k < \frac{n}{2}\right) \\
&\leq q^{n-1} n^k \\
&= \exp((n-1)\ln q + k \ln n) \\
&= \exp\left((n-1)\ln q + \left(\frac{-\ln q}{2}\frac{n}{\ln n}\right)\ln n\right) \\
&= \exp\left(\left(\frac{n}{2} - 1\right)\ln q\right) \\
&\to 0 && (\text{as } n \to \infty).
\end{aligned}$$

Hence, $\kappa(G)$ is asymptotically almost surely greater than $\frac{-\ln q}{2}\frac{n}{\ln n}$.

Taken together, we see that $\alpha(G)$ is asymptotically almost surely less than $\kappa(G)$, and so conclude that G is asymptotically almost surely Hamiltonian. □

2 Problem 2

Proposition 2.1. *Suppose G and H are Hamiltonian. Show that $G \square H$ is Hamiltonian as well.*

Proof. Let $u_1 \cdots u_m u_1$ and $v_1 \cdots v_n v_1$ be Hamiltonian cycles in G and H, respectively. We consider two cases.

Suppose at least one of m or n is even. Due to the symmetry of the Cartesian product, we can assume, without loss of generality, that n is even. We now construct a Hamiltonian cycle in $G \square H$. It follows from the definition of the Cartesian product and the existence of a Hamiltonian cycle (and so a Hamiltonian path) in G that

$$(u_1, v_1) \sim (u_2, v_1) \sim \cdots \sim (u_m, v_1).$$

Now, using the Hamiltonian path in H, we see that

$$(u_m, v_1) \sim (u_m, v_2).$$

We now find ourselves in a new "copy" of G in $G\Box H$. We next traverse the Hamiltonian path of G in the reverse order. That is, we observe that

$$(u_m, v_2) \sim (u_{m-1}, v_2) \sim \cdots \sim (u_1, v_2).$$

As before, we can take a single step on the Hamiltonian path in H. That is,

$$(u_1, v_2) \sim (u_1, v_3).$$

Continuing in this fashion, we can build a Hamiltonian path from (u_1, v_1) to (u_1, v_n) (since n is even, we know the first coordinate of the terminal vertex will be u_1 rather than u_m). Since $v_n \sim v_1$ in H, we have that $(u_1, v_n) \sim (u_1, v_1)$, thus completing the Hamiltonian cycle in $G\Box H$.

Suppose now that both m and n are odd and assume. By the symmetry of the Cartesian product, we can assume, without loss of generality, that $m \leq n$. We now construct a Hamiltonian cycle in $G\Box H$, beginning with the following string of adjacencies:

$$\begin{gathered}
(u_1, v_1) \sim (u_2, v_1) \sim \cdots \sim (u_m, v_1) \sim \\
(u_m, v_2) \sim (u_1, v_2) \sim \cdots \sim (u_{m-1}, v_2) \sim \\
\vdots \\
(u_2, v_m) \sim (u_3, v_m) \sim \cdots \sim (u_1, v_m).
\end{gathered}$$

Whereas in the previous construction we alternately traverse the Hamiltonian path in each copy of G in increasing and decreasing order with regard to the indices, here we only traverse the paths in increasing order. As a result, we omit a different edge in each copy of G. Now, if $m = n$, we simply close the cycle with the edge $(u_1, v_m)(u_1, v_1)$. Otherwise, we extend the path in the following way:

$$\begin{gathered}
(u_1, v_m) \sim \\
(u_1, v_{m+1}) \sim (u_2, v_{m+1}) \sim \cdots \sim (u_n, v_{m+1}) \sim \\
(u_n, v_{m+2}) \sim (u_{n-1}, v_{m+2}) \sim \cdots \sim (u_1, v_{m+2}) \sim \\
\vdots \\
(u_n, v_n) \sim (u_{n-1}, v_n) \sim \cdots \sim (u_1, v_n).
\end{gathered}$$

In this extension, we alternate an increasing and decreasing traversal of the vertices so that the same edge is omitted from each copy of G. Since both m and n are odd, $n - m$ is even, and so we are guaranteed that this extended path terminates at (u_1, v_n) rather than (u_n, v_n). Finally, adding the edge $(u_1, v_n)(u_1, v_1)$ completes the desired Hamiltonian cycle. $\square$

3 Problem 3

Definition 3.1. *A subgraph $H \subset G$ is said to* edge-dominate *G if $||G - H|| = 0$.*

Proposition 3.2. *Suppose $\|G\| \geq 3$. The line graph $L(G)$ is Hamiltonian if and only if G contains a connected, edge-dominating, even subgraph.*

Proof. ($\Rightarrow$) Suppose that $L(G)$ possesses a Hamiltonian cycle $e_1 e_2 \cdots e_m e_1$. For $1 \leq i \leq m - 1$, let v_i denote the vertex in G witnessing the incidence of e_i and e_{i+1} in G (note that the v_i need not be distinct). In addition, let v_m denote the vertex of G witnessing the adjacency of e_m and e_1. Finally, let $C = v_1 v_2 \cdots v_m v_1$, where non-distinct consecutive vertices are simply regarded as a redundancy (i.e. we do not regard this repetition as indicative of a loop). We see that C is indeed a cycle of G, as any v_i and v_{i+1} are either non-distinct or are adjacent in G (they are connected by the edge e_{i+1}). Similarly, v_m is adjacent to v_1, as evidenced by the edge e_1. By construction, every edge of G is adjacent to some vertex of C, and so C is the desired connected, edge-dominating, even subgraph of G.

($\Leftarrow$) Suppose that G contains a connected, edge-dominating, even subgraph H. As H is even, it possesses an Euler tour $v_1 v_2 \cdots v_\ell v_1$. For $1 \leq i \leq \ell - 1$, denote by e_i the edge of the tour between v_i and v_{i+1}. In addition, let e_ℓ denote the edge of the tour between v_ℓ and v_1.

Now, we have that $C = e_1 e_2 \cdots e_\ell e_1$ is a cycle in $L(G)$. As H edge-dominates G, any edge e of $E(G) - E(H)$ must have some v_i as one of its endvertices, and so e is incident to at least e_{i-1} and e_i. Hence, we can extend C to a larger cycle by inserting e between e_{i-1} and e_i. Continuing in this fashion, we can extend C so as to include all of $E(G)$, and so C becomes a Hamiltonian cycle of G. $\square$

4 Problem 5

Definition 4.1. *A graph G is* uniquely k-edge-colorable *if any two k-edge-colorings of G induce the same partition of $E(G)$.*

Proposition 4.2. *Every uniquely 3-edge-colorable cubic graph is Hamiltonian.*

Proof. Let G be a uniquely 3-edge-colorable cubic graph. Since G is cubic, every vertex is incident to exactly one edge frome each of the three color classes. Hence, each color class induces a perfect matching of G. Let H be the graph whose edge set is union of the edges of any two color classes. Evidently, H is a 2-regular subgraph of G, and so is the union of disjoint cycles. Suppose, for the purpose of contradiction, that H is comprised of more than one cycle. As the cycles are disjoint, we may freely exchange the colors of one of the cycles, thus contradicting the unique 3-colorability of G. Hence, it must be that H is a single cycle, and so is in fact a Hamiltonian cycle of G. □

Remark 4.3. *The above argument could be carried out for all* $\binom{3}{2} = 3$ *pairs of color classes, and so we have actually proven that G contains* three *Hamiltonian cycles.*

5 Problem 6

Proposition 5.1. *Let* $\epsilon > 0$ *and* $p = p(n) > 0$*, and let* $r \geq \frac{1}{p}(1+\epsilon)(2\ln n)$ *be an integer-valued function of* n*. Almost no graph in* $\mathcal{G}(n,p)$ *contains* r *independent vertices.*

Proof. Let $q = 1-p$. For all integers n, r with $n \geq r \geq 2$ and all $G \in \mathcal{G}(n,p)$, we have

$$\begin{aligned}
P[\alpha(G) \geq r] &\leq \binom{n}{r} q^{\binom{r}{2}} \\
&\leq n^r q^{\binom{r}{2}} \\
&= \left(nq^{\frac{1}{2}(r-1)}\right)^r \\
&\leq \left(ne^{\frac{-1}{2}p(r-1)}\right)^r \\
&= \left(ne^{\frac{-1}{2}pr}\right)^{r-1} \\
&\leq \left(ne^{\frac{-1}{2}p\left[\frac{1}{p}(1+\epsilon)(2\ln n)\right]}\right)^{r-1} \\
&= \left(ne^{-(1+\epsilon)\ln n}\right)^{r-1} \\
&= \left(nn^{-(1+\epsilon)}\right)^{r-1} \\
&= \left(n^{-\epsilon}\right)^{r-1}
\end{aligned}$$

$$= \left(n^{1-r}\right)^{\epsilon}.$$

As $\epsilon > 0$ is constant and $p \in (0, 1]$, we see that $n^{1-r} \to 0$ as $n \to \infty$. That is, almost no graph in $\mathcal{G}(n, p)$ contains r independent vertices. $\square$

6 Problem 7

Proposition 6.1. *For every* $0 < \epsilon \leq 1$ *and* $p = (1-\epsilon)(\ln n)n^{-1}$*, almost every* $G \in \mathcal{G}(n, p)$ *has an isolated vertex.*

Lemma 6.2. *If* $q = 1 - p$*, then*

$$nq^{n-1} \to \infty \text{ as } n \to \infty.$$

Proof. Observe first that

$$\begin{aligned} q^{n-1} &= \left(1 + \frac{(\epsilon - 1)\ln n}{n}\right)^{n-1} \\ &\geq \left(1 + \frac{(\epsilon - 1)\ln n}{n}\right)^{n} \\ &\to e^{(\epsilon-1)\ln n} && \text{(as } n \to \infty\text{)} \\ &= n^{\epsilon-1}. \end{aligned}$$

Hence,

$$\begin{aligned} \lim_{n\to\infty} nq^{n-1} &\geq \lim_{n\to\infty} nn^{\epsilon-1} \\ &= \lim_{n\to\infty} n^{\epsilon} \\ &= \infty. \end{aligned}$$

$\square$

Proof. (of 6.1)

Let $G \in \mathcal{G}(n, p)$ and let $X(G)$ denote the number of isolated vertices in G. For each vertex $v \in G$, define $X_v(G)$ to be the indicator random variable that equals 1 whenever v is isolated and 0 otherwise. It follows that

$$\begin{aligned} E(X) &= E\left(\sum_{v\in V(G)} X_v\right) \\ &= \sum_{v\in V(G)} E(X_v) && \text{(by linearity of expectation)} \end{aligned}$$

$$
\begin{aligned}
&= \sum_{v \in V(G)} q^{n-1} && \text{(where } q = 1-p\text{)} \\
&= nq^{n-1}.
\end{aligned}
$$

Since $q^{n-1} \to 1$ as $n \to \infty$, $E(X) > 0$ for large n. Now,

$$
\begin{aligned}
E(X^2) &= E\left(\sum_{(v,w) \in V(G)^2} X_v X_w \right) \\
&= \sum_{(v,w) \in V(G)^2} E(X_v X_w) && \text{(by linearity of expectation)} \\
&= \sum_{v} E(X_v) + \sum_{v \neq w} E(X_v X_w) \\
&= E(X) + n(n-1)q^{n-1}q^{n-2} \\
&\leq E(X) + n^2 q^{2n-3}.
\end{aligned}
$$

Finally,

$$
\begin{aligned}
\frac{E(X^2) - E(X)^2}{E(X)^2} &\leq \frac{E(X) + n^2 q^{2n-3} - E(X)^2}{E(X)^2} \\
&= \frac{1}{E(X)} + \frac{n^2 q^{2n-3}}{E(X)^2} - 1 \\
&= \frac{1}{nq^{n-1}} + \frac{n^2 q^{2n-3}}{(nq^{n-1})^2} - 1 \\
&= \frac{1}{nq^{n-1}} + \frac{1}{q} - 1 \\
&\to 0 + 1 - 1 && \text{(as } n \to \infty\text{)} \\
&= 0.
\end{aligned}
$$

Therefore, $X(G) > 0$ asymptotically almost surely. That is, G almost surely contains an isolated vertex. ☐

Proposition 6.3. *There is a probability $p = p(n)$ such that almost every $G \in \mathcal{G}(n,p)$ is disconnected but the expected number of spanning trees of G tends to infinity as $n \to \infty$.*

Proof. Let $p = (1-\epsilon)(\ln n)n^{-1}$ with $\epsilon \in (0,1)$ and let $G \in \mathcal{G}(n,p)$. By the previous result, G contains an isolated vertex asymptotically almost surely (and so is disconnected). Let now $\mathcal{T}$ denote the collection of all possible spanning trees of G. For a given $T \in \mathcal{T}$, tet $X_T(G)$ be the random variable indicating that T is indeed a spanning of G. Finally, let $X(G)$ denote the total number of spanning trees of G. It follows that

$$\begin{aligned}
E(X) &= E\left(\sum_{T\in\mathcal{T}} X_T\right) \\
&= \sum_{T\in\mathcal{T}} E(X_T) && \text{(by linearity of expectation)} \\
&= \sum_{T\in\mathcal{T}} p^{n-1} \\
&= n^{n-2}p^{n-1} \\
&= n^{n-2}((1-\epsilon)(\ln n)n^{-1})^{n-1} \\
&= n((1-\epsilon)\ln n)^{n-1} \\
&\to \infty && \text{(as } n \to \infty\text{).}
\end{aligned}$$

Hence, while G is aysmptotically almost surely disconnect, the expected number of spanning trees in G tends to infinity as $n \to \infty$. ☐

7 Problem 8

Proposition 7.1. *The function $t(n) = n^{-1}$ is a threshold function for the property of containing* any *cycle.*

Proof. Let $G \in \mathcal{G}(n, p)$.

Suppose first that $np \to \infty$ as $n \to \infty$. Since n^{-1} is a threshold function for containing a k-cycle for any fixed k, G contains a k-cycle (and so contains *some* cycle) almost surely.

Suppose now that $np \to 0$ as $n \to \infty$. Let $X_k(G)$ denote the number of k-cycles in G and $X(G)$ denote the total number of cycles in G. It follows that

$$\begin{aligned}
\mathrm{P}\left[X \geq 1\right] &\leq E(X) && \text{(by Markov's Inequality)} \\
&= E\left(\sum_{k=3}^{n} X_k(G)\right) \\
&= \sum_{k=3}^{n} E(X_k(G)) && \text{(by linearity of expectation)}
\end{aligned}$$

$$
\begin{aligned}
&= \sum_{k=3}^{n} \frac{(n)_k p^k}{2k} \\
&\leq \sum_{k=3}^{n} (np)^k \\
&\leq \sum_{k=0}^{\infty} (np)^k \\
&= \frac{np}{1-np}.
\end{aligned}
$$

Hence, $\mathrm{P}\left[X \geq 1\right] \leq \frac{np}{1-np} \to 0$ as $n \to \infty$, and so G almost surely contains no cycle.

Taken together, we see that n^{-1} is indeed a threshold function for the property of containing any cycle. □