Final Exam

MATH 776, Fall 2009, Mohr

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1 Problem 1

Definition 1.1. *The family $\mathcal{C}$ of cographs consists of (1) K^1, (2) the complement of any cograph, (3) the disjoint union of any two cographs.*

Lemma 1.2. *A cograph can be reduced to a set of isolated vertices by iterated complementation of its components.*

Proof. Let C be a component of a cograph G. If $\overline{C}$ is connected, then both C and $\overline{C}$ are atomic cographs in the sense that they are not the complement of the disjoint union of smaller cographs. The only graph that does not arise in this way is the given graph K^1, and so $C = \overline{C} = K^1$.
It follows that, if $C \neq K^1$, then $\overline{C}$ is disconnected. Hence, we can proceed inductively by taking the complement of the components of $\overline{C}$ until we are left with only copies of K^1. □

Lemma 1.3. *The graph G contains an induced P^3 if and only if $\overline{G}$ does.*

Proof. Let G contain an induced P^3. There are vertices v_1, v_2, v_3, v_4 in G with $v_1 \sim v_2$, $v_2 \sim v_3$, $v_3 \sim v_4$, and all other pairs nonadjacent. Hence, in $\overline{G}$ we have $v_2 \sim v_4$, $v_4 \sim v_1$, $v_1 \sim v_3$, and all other pairs nonadjacent. That is, $\overline{G}$ contains an induced P^3.
The reverse implication follows from the fact that $\overline{\overline{G}} = G$. □

Lemma 1.4. *The disjoint union of perfect graphs is perfect.*

Proof. Let G and H be disjoint perfect graphs. We have immediately that

$$\begin{aligned} \chi(G \cup H) &= \max\{\chi(G), \chi(H)\} && \text{(since } G \text{ and } H \text{ are disjoint)} \\ &= \max\{\omega(G), \omega(H)\} && \text{(since } G \text{ and } H \text{ are perfect)} \\ &= \omega(G \cup H) && \text{(since } G \text{ and } H \text{ are disjoint).} \end{aligned}$$

□

Lemma 1.5. *The complement of a perfect graph is perfect.*

Proof. (partial) Let G be a perfect graph. A result of Lovász gives that, for all induced subgraphs H of G, $|H| \leq \alpha(H)\omega(H)$. Now, for any induced subgraph $\overline{H_0}$ of $\overline{G}$, we have

$$\begin{aligned} |\overline{H_0}| &= |H_0| \\ &\leq \alpha(H_0)\omega(H_0) && \text{(since } G \text{ is perfect)} \\ &= \omega(\overline{H_0})\alpha(\overline{H_0}). \end{aligned}$$

Hence, $\overline{G}$ is perfect. □

Proposition 1.6. *The cographs are precisely those graphs which contain no induced P^3.*

Proof. Let $\mathcal{P}$ denote the family of graphs containing no induced P^3. We show that $\mathcal{C} = \mathcal{P}$.
($\subseteq$) Let $G \in \mathcal{C}$. By 1.2, G can be reduced to a set of isolated vertices by iterated complementation of its compnents. Combining this fact with 1.3, it follows that G contains no induced P^3.
($\supseteq$) (I attempted to emulate the induction proof we sketched yesterday, but every time I tried to force G to have an induced P_3, I seemed to find a way to avoid it without achieving the desired conclusion.) □

Proposition 1.7. *Cographs are perfect.*

Proof. (by strong induction on $|G|$)
If $|G| = 1$, then $G = K^1$, which is perfect.
Suppose now that $|G| = n$ and all cographs on at most $n-1$ vertices are perfect. If G arises as the disjoint union of two smaller (and so perfect) cographs, then G is perfect by 1.4. Otherwise, it must be that $\overline{G}$ is the disjoint union of two smaller cographs G_1 and G_2 by 1.2. By the inductive hypothesis, each of G_1 and G_2 is perfect, and so their union is perfect by 1.4. Finally, 1.5 gives that G itself is perfect. □

2 Problem 2

Lemma 2.1. *The chromatic polynomial of the cycle on n vertices is $(\lambda-1)^n + (-1)^n(\lambda-1)$.*

Proof. (by induction on n)
For $n = 3$, observe that C_3 is the same as K^3, giving $\lambda(\lambda-1)(\lambda-2) = \lambda^3 - 3\lambda^2 + 2\lambda$ colorings. Comparing this with the proposed formula, we see

$$\begin{aligned} p_{C_3}(\lambda) &= (\lambda-1)^3 + (-1)^3(\lambda-1) \\ &= (\lambda^3 - 3\lambda^2 + 3\lambda - 1) - (\lambda-1) \\ &= \lambda^3 - 3\lambda^2 + 2\lambda, \end{aligned}$$

and so the formula hold in the case of $n = 3$.
For $n \geq 4$, observe that

$$\begin{aligned} p_{C_n}(\lambda) &= p_{C_n - e}(\lambda) - p_{C_n/e}(\lambda) && \text{(some } e \in E(C_n)) \\ &= p_{P_{n-1}}(\lambda) - p_{C_{n-1}}(\lambda) \\ &= \lambda(\lambda-1)^{n-1} - \left[(\lambda-1)^{n-1} + (-1)^{n-1}(\lambda-1)\right] \\ &= \lambda(\lambda-1)^{n-1} - (\lambda-1)^{n-1} - (-1)^{n-1}(\lambda-1) \\ &= \lambda(\lambda-1)^{n-1} - (\lambda-1)^{n-1} + (-1)^n(\lambda-1) \\ &= (\lambda-1)^{n-1}(\lambda-1) + (-1)^n(\lambda-1) \\ &= (\lambda-1)^n + (-1)^n(\lambda-1), \end{aligned}$$

as desired. □

Proposition 2.2. *The chromatic polynomial of the wheel graph $K^1 * C_n$ is $\lambda\left[(\lambda-2)^{n-1} + (-1)^{n-1}(\lambda-2)\right]$.*

Proof. Let v be the central vertex of the wheel graph W_n. First, choose any of the available λ colors to color v. Since v is adjacent to all other vertices, the chosen color cannot be used again. Hence, it remains to color $W_n - v$ (i.e. C_n) using $\lambda - 1$ colors. Making use of 2.1, we have

$$\begin{aligned} p_{W_n}(\lambda) &= \lambda p_{C_n}(\lambda-1) \\ &= \lambda\left[(\lambda-2)^n + (-1)^n(\lambda-2)\right], \end{aligned}$$

as desired. □

Proposition 2.3. *The chromatic polynomial of $K_{2,s}$ is $\lambda(\lambda-1)\left[(\lambda-1)^{s-1} + (\lambda-2)^s\right]$.*

Proof. Let v and w be the vertices in the partite set of size 2. We consider two cases.
If v and w are colored the same, then we choose a single color of the available λ colors. As both v and w are adjacent to all of the remaining s vertices, we color using the remaining $\lambda - 1$ colors. As these s vertices are independent, we can color freely, and so arrive at $\lambda(\lambda-1)^s$ colorings for this case.
If v and w are colored differently, then there are $\lambda(\lambda-1)$ ways to color them. As before, we use the remaining $\lambda - 2$ colors to color the remaining s vertices freely, giving $\lambda(\lambda-1)(\lambda-2)^s$ colorings for this case.
Taken together, we have that

$$\begin{aligned} p_{K_{2,s}}(\lambda) &= \lambda(\lambda-1)^s + \lambda(\lambda-1)(\lambda-2)^s \\ &= \lambda(\lambda-1)\left[(\lambda-1)^{s-1} + (\lambda-2)^s\right], \end{aligned}$$

as desired. □

3 Problem 3

Definition 3.1. *An interval representation of a graph $G = (V, W)$ is a family $\mathcal{I}$ of intervals of the real line and a bijection $\phi : V \to \mathcal{I}$ so that $vw \in E$ if and only if $\phi(v) \cap \phi(w) \neq \emptyset$. Such a representation is "proper" if no element of $\mathcal{I}$ contains another.*

Proposition 3.2. *An interval graph is claw-free (no induced $K_{1,3}$) if and only if it has a proper representation.*

Proof. ($\Rightarrow$) We proceeed by establishing the contrapositive. To that end, let G be an interval graph having no proper representation. For all representations, there are intervals I_1 and I_2 such that $I_1 \subseteq I_2$. Furthermore, there are intervals I_3 and I_4 with I_1 properly between them (and so $I_3 \cap I_4 = \emptyset$) and such that $I_2 \cap I_3 \neq \emptyset$ but $I_2 \cap I_1 = \emptyset$ and $I_4 \cap I_3 \neq \emptyset$ but $I_4 \cap I_1 = \emptyset$. Were this not the case, then I_1 could be freely extended to the left or the right so as not to be contained in I_2, thus giving a proper representation. Now, let v_j be the vertex in G corresponding to I_j. We have that v_2 is adjacent to v_1, v_3, and v_4, while the latter three vertices are themselves pairwise nonadjacent. Hence, G is not claw-free.
($\Leftarrow$) We proceed by establishing the contrapositive. To that end, let G be an interval graph that is not claw-free. There are vertices v_1, v_2, v_3, and v_4 such that, without loss of generality, v_1 is adjacent to all of v_2, v_3, and v_4, while the latter three vertices are themselves pairwise nonadjacent. Letting I_j be the interval corresponding to the vertex v_j shows that I_1 intersects nonemptily with each of I_2, I_3, and I_4, while the latter three intervals are themselves pairwise disjoint. This happens only if one of I_2, I_3, or I_4 is contained in I_1. Hence, G has no proper representation. □

4 Problem 4

Proposition 4.1. *The complement of a bipartite graph is perfect.*

Proof. Let G be such that $\overline{G}$ is bipartite. It suffices to show that G is perfect, as any induced subgraph of G is also the complement of a bipartite graph (specifically, the complement of some induced subgraph of $\overline{G}$).
Observe first that $\chi(G) \geq \omega(G)$. Now, the color classes of any $\chi(G)$-coloring contain at most two vertices (else $\overline{G}$ contains a clique of size larger than 2, and so is not bipartite). Let k_1 denote the number of color classes containing a single vertex (and V_1 denote the set of these vertices) and k_2 denote the number color classes containing two vertices. Evidently, those vertices that belong to a color class of size 1 form a clique of size k_1 in G (and so form an independent set of size k_1 in $\overline{G}$). Next, observe that by choosing a single vertex from each color class of size 2, we construct a minimum vertex cover of $\overline{G}$, and so the maximum size of a matching in $\overline{G}$ is k_2 by König's Theorem. We can deduce that, for every edge uv in a maximum matching of $\overline{G}$, one of u or v is incident to V_1.

Suppose this is not the case. That is, $u \sim w_1$ and $v \sim w_2$ for some $w_1, w_2 \in V_1$. Since $\overline{G}$ is bipartite, $w_1 \neq w_2$. Hence, we can construct a strictly larger matching by discarding the edge uv and adding the edges uw_1 and vw_2 - a contradiction. Thus, we have that $\overline{G}$ contains an independent set of size at least $k_1 + k_2 = \chi(G)$, so $\alpha(\overline{G}) \geq \chi(G)$. That is, $\omega(G) \geq \chi(G)$. Therefore, $\chi(G) = \omega(G)$ (i.e. G is perfect). □

5 Problem 5

Lemma 5.1. *Every chordal graph has a simplicial vertex.*

Proof. We know that every chordal graph can be constructed recursively by pasting along complete subgraphs starting with complete graphs. In a complete graph, every vertex is simplicial. Now, let G be a chordal graph obtained by pasting two chordal graphs G_1 and G_2 as described above. By the inductive hypothesis, each of G_1 and G_2 has a simplicial vertex. As $G_1 \subset G$, it follows that G also has a simplicial vertex. □

Lemma 5.2. *If v is a simplicial vertex in a graph G whose neighborhood is a clique of size k, then $p_G(\lambda) = (\lambda - k)p_{G-v}(\lambda)$.*

Proof. There are $p_{G-v}(\lambda)$ ways to color $G - v$ using λ colors. Since v is adjacent to a k-clique, we are left with $\lambda - k$ ways to color v. □

Proposition 5.3. *The chromatic polynomial of a chordal graph has only integer roots.*

Proof. Let G be a chordal graph. By 5.1, G has a simplicial vertex v. Let v be adjacent to a clique of size k in G. By 5.2, we have that $p_G(\lambda) = (\lambda - r)p_{G-v}(\lambda)$. Now, as $G - v$ is chordal, it possesses a simplicial vertex. Proceeding inductively, we can factor p_G into linear factors having integer roots. □

6 Problem 6

Proposition 6.1. *Given two vertex colorings f and g of G with $D = \mathrm{col}(G) + 1$ colors, there exists a sequence of vertex D-colorings $f = c_1, c_2, \ldots, c_{k-1}, c_k = g$ so that c_j differs from c_{j+1} at exactly one vertex for each $j = 1, \ldots, k-1$.*

Proof. (by induction on $|G|$)
If $|G| = 1$, then $\mathrm{col}(G) = 1$, and so there are only two possible colorings of G on $D = 2$ colors. As G has no edges, we can switch from one coloring to another by simply recoloring the single vertex in G.
Let now $|G| = n$ and let v be a vertex of G of least degree. By the inductive hypothesis, there is a sequence of colorings of $G - v$ having the desired properties. Given a coloring of G, proceed to recolor it using the same sequence. This will satisfactorily recolor $G - v$ except in the case where a neighbor of v is assigned the same color as v. As the neighborhood of v is guaranteed to use no more than $\mathrm{col}(G)$ colors at any given stage of the recoloring, it follows that we can interrupt the recoloring of $G - v$, recolor v with an unused color, and proceed with the recoloring of $G - v$. Once $G - v$ is colored appropriately, we finish by assigning the correct color to v which, by the same argument, can always be done. □

7 Problem 7

Proposition 7.1. *Suppose G and H are nonempty and connected. $G \Box H$ is planar if and only if either (a) one of G or H is K^1 and the other is planar, (b) one of G or H is K^2 and the other is outerplanar, or (c) one of G or H is P^r for some $r \geq 2$ and the other is either P^s for some $s \geq 2$ or a cycle.*

Proof. (⇒) We first make a couple of observations which will allow us to characterize which G and H give planar $G\Box H$.
First, observe that one of G or H must be a tree. If not, then both G and H contain a C_3 minor. Denote the vertices of the C_3 minor in G by u_1, u_2, and u_3 and denote the vertices of the C_3 minor in H by v_1, v_2, and v_3. In $C_3\Box C_3$, we have the $K_{3,3}$ minor with partite sets $\{(u_1,v_1),(u_2,v_2),(u_3,v_3)\}$ and $\{\{(u_1,v_2),(u_1,v_3)\},\{(u_2,v_1),(u_2,v_3)\},\{(u_3,v_1),(u_3,v_2)\}\}$. Hence, $G\Box H$ is not planar.
Next, observe that if one of G or H has maximum degree 2, then the other has maximum degree at most 2. Suppose this is not the case. Let G have maximum degree 3 and H have maximum degree 2. We have that G contains a $K_{1,3}$ subgraph and H contains a P^2 subgraph. Denote the vertices of the $K_{1,3}$ in G by u_1, u_2, u_3, and u_4, with u_1 being the central vertex. Denote the vertices of the P^2 in H by v_1, v_2, and v_3. Now, in $G\Box H$, we have the $K_{3,3}$ minor with partite sets $\{(u_1,v_1),(u_1,v_2),(u_1,v_3)\}$ and $\{\{(u_2,v_1),(u_2,v_2),(u_2,v_3)\},\{(u_3,v_1),(u_3,v_2),(u_3,v_3)\},\{(u_4,v_1),(u_4,v_2),(u_4,v_3)\}\}$. Hence, $G\Box H$ is not planar.
Now, let H be a tree with maximum degree at most 2. We consider all cases.
If H has maximum degree 0, then $H = K^1$. Evidently, $G\Box H = G$. Hence, $G\Box H$ is planar if and only if G is planar.
If H has maximum degree 1, then $H = K^2$. Hence, $G\Box H$ can be viewed as two copies of G in which an additional edge is added between corresponding vertices. Now, if G is not outerplanar, then every drawing has some vertex which is not on the frontier of the unbounded face in the plane. Hence, this vertex in one copy cannot be connected to its corresponding vertex in the other copy without inducing a crossing. Hence, G must be outerplanar.
If H has maximum degree 2 and is a tree, then $H = P^r$ for some $r \geq 2$. Since G must have maximum degree at most 2, it follows that G can be P^s for some $s \geq 2$ or a cycle. Indeed, both of these possibilities result in planar $G\Box H$ (see below).
(⇐) We show that each of the listed choices for G and H indeed results in planar $G\Box H$.
If G is planar and H is K^1, then $G\Box H \simeq G$, which is planar.
If G is outerplanar and H is K^2, then $G\Box H$ can be viewed as 2 copies of G with edges added between corresponding vertices in the copies. Project one copy of G to the sphere. Since G is outerplanar, it can be drawn so that all its vertices lie on some disk and, furthermore, all its edges lie within this disk. Now, an appropriate homeomophism can redraw G so that all its edges lie *outside* this disk. Projecting back to the plane, we can nest the original copy of G inside this "inverted" copy of G and add edges between corresponding vertices without inducing a crossing. Hence, $G\Box H$ is planar.
If G is P^s and H is P^r for some $r, s \geq 2$, then $G\Box H$ is an $r \times s$ grid, which is planar.
If G is P^s for some $s \geq 2$ and H is a cycle, then $G\Box H$ can be realized as s nested copies of H. In addition, an edge is introduced between corresponding vertices of a cycle and the one nested immediately inside it. By this construction, we see that $G\Box H$ is planar. □

8 Problem 8

Definition 8.1. *A graph G is called χ-unique if $p_G = p_H$ implies $G \simeq H$.*

Proposition 8.2. *$K_{n,n}$ is χ-unique.*

Proof. Let G be such that $p_G = p_{K_{n,n}}$. We have that

$$\begin{aligned} |G| &= \deg p_G \\ &= \deg p_{K_{n,n}} \\ &= |K_{n,n}| \\ &= 2n. \end{aligned}$$

Next, observe that

$$\begin{aligned} K_{n,n} \text{ is bipartite} &\Rightarrow p_{K_{n,n}}(2) \neq 0 \\ &\Rightarrow p_G(2) \neq 0 \end{aligned}$$

$$\Rightarrow G \text{ is bipartite.}$$

Finally, by Whitney's Broken Circuit Theorem,

$$\begin{aligned} ||G|| &= [x^{n-1}]p_G \\ &= [x^{n-1}]p_{K_{n,n}} \\ &= n^2. \end{aligned}$$

Now, let G have partite sets of size k and $2n-k$. It follows that $||G|| = k(2n-k)$, which equals n^2 only when $k = n$. Hence, G is bipartite with partite sets both of size n. Since we previously determined that $||G|| = n^2$, it follows that $G \simeq K_{n,n}$. □