

Priestley's Representation Theorem

Math 743 - Spring 2010

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1 Preliminaries

Theorem 1.1. (*Prime Ideal Theorem*) Let \mathbf{L} be a distributive lattice and I be an ideal of \mathbf{L} . For any element $a \in L$ with $a \notin I$, there exists a prime ideal containing I and missing a .

Theorem 1.2. (*Compactness Theorem*) Let \mathbf{L} be a distributive lattice. Given subsets S and T of L with the property that every prime ideal containing S hits T , there exist finite subsets S_0 and T_0 of S and T , respectively, such that every prime ideal containing S_0 hits T_0 .

Theorem 1.3. (*Alexander's Subbase Theorem*) A topological space is compact if and only if every subbasic open cover of the space admits a finite subcover.

Theorem 1.4. A collection \mathcal{B} is a base for a topology on X if and only if

1. $X = \bigcup_{B \in \mathcal{B}} B$
2. whenever $B_1, B_2 \in \mathcal{B}$ with $x \in B_1 \cap B_2$, there is some $B_3 \in \mathcal{B}$ with $x \in B_3 \subseteq B_1 \cap B_2$.

2 Priestley's Representation Theorem

Definition 2.1. Let Y be a set partially ordered by \leq with least element 0 and greatest element 1 , and let τ be a compact topology on Y . The system $\langle Y, \leq, 0, 1, \tau \rangle$ is called a Priestley space provided that, for all $x, y \in Y$, there is a clopen downset containing y but not x whenever $x \not\leq y$.

Theorem 2.2. (*Priestley's Representation Theorem*)

Let \mathbf{L} be a distributive lattice and \mathcal{P} be the collection of prime ideals of \mathbf{L} ordered by inclusion. (Hence, \mathcal{P} has least element \emptyset and greatest element L .) Let τ be the topology generated by the subbasis

$$\{e(a) \mid a \in L\} \cup \{e(a)^c \mid a \in L\},$$

where $e(a)$ denotes the set of elements of \mathcal{P} missing a .

1. $\langle \mathcal{P}, \subseteq, \emptyset, \mathbf{L}, \tau \rangle$ is a Priestley space.
2. $\{e(a) \mid a \in L\}$ is exactly the set of proper, nonempty, clopen downsets of \mathcal{P} .

Proof. Evidently, \mathcal{P} is partially ordered by \subseteq with least element \emptyset and greatest element L . To verify that $\langle \mathcal{P}, \subseteq, \emptyset, \mathbf{L}, \tau \rangle$ is a Priestley space, it remains to show that τ is compact and that, for all $I, J \in \mathcal{P}$ with $I \not\subseteq J$, there is a clopen downset containing J but not I .

To show that τ is compact, let \mathcal{C} be a subbasic open cover of \mathcal{P} (by Alexander's Subbase Theorem, a subbasic open cover is sufficient to demonstrate compactness). Define the sets $S = \{a \mid e(a) \in \mathcal{C}\}$ and $T = \{a \mid e(a)^c \in \mathcal{C}\}$. Note that both S and T are nonempty, since $\emptyset \notin e(a)^c$ for any $a \in L$ and $L \notin e(a)$ for any $a \in L$. Now, consider any ideal I containing S and suppose, for the purpose of contradiction, that $I \cap T = \emptyset$. By the Prime Ideal Theorem, we can find a prime ideal P with $I \subseteq P$ and $P \cap T = \emptyset$. By construction, $S \subseteq P$, and so $P \notin e(a)$ for any $a \in S$. Similarly, since $P \cap T = \emptyset$, $P \notin e(a)^c$ for any $a \in T$. We conclude that \mathcal{C} does not cover \mathcal{P} , which is a contradiction. Hence, it must be that $I \cap T \neq \emptyset$. In other words, every ideal containing S must intersect T . By the Compactness Theorem, we can find finite subsets S_0 and T_0 of S and T , respectively, such that every prime ideal containing S_0 intersects T_0 . Let \mathcal{C}' denote the collection $\bigcup_{a \in S_0} e(a) \cup \bigcup_{a \in T_0} e(a)^c$. Evidently, \mathcal{C}' is a finite subcollection of \mathcal{C} . We claim that it also covers \mathcal{P} , thus establishing the compactness of τ . To that end, let P be a prime ideal. If P contains S_0 , then by the previous remarks, P intersects T_0 , and so $P \in e(a)^c$ for some $a \in T_0$. If P does not contain S_0 , then P misses some element of $a \in S_0$, and so $P \in e(a)$.

For the remaining claim, let $I, J \in \mathcal{P}$ with $I \not\subseteq J$. Choose any element $a \in I$ with $a \notin J$. Evidently, $J \in e(a)$, which is clopen, since $e(a)^c$ is an element of the subbase. To see that $e(a)$ is also a downset, let K be any prime

ideal contained in J . Since $a \notin J$, it follows that $a \notin K$, and so $K \in e(a)$, as desired.

We show next that $\{e(a) \mid a \in L\}$ is exactly the set of proper, nonempty, clopen downsets of \mathcal{P} . We have just shown that any $e(a)$ is a clopen downset of \mathcal{P} . Furthermore, each $e(a)$ is nonempty (as $\emptyset \in e(a)$ for all $a \in L$) and proper (as $L \notin e(a)$ for any $a \in L$). Hence, the set $\{e(a) \mid a \in L\}$ is contained in the set of proper, nonempty, clopen downsets of \mathcal{P} .

Before proving the other inclusion, we first claim that the collection $\{e(a) \mid a \in L\}$ is a base for the subspace $\mathcal{P} \setminus L$. Indeed,

$$\begin{aligned} P \in \mathcal{P} \setminus L &\Leftrightarrow P \text{ is a proper prime ideal of } L \\ &\Leftrightarrow \text{there exists } b \in L \text{ with } b \notin P \\ &\Leftrightarrow P \in e(b) \text{ for some } b \in L \\ &\Leftrightarrow P \in \bigcup_{a \in L} e(a), \end{aligned}$$

and hence $\mathcal{P} \setminus L = \bigcup_{a \in L} e(a)$. Next, suppose that $P \in e(b) \cap e(c)$ for some $b, c \in L$. By definition, P is an ideal missing both b and c , and hence P misses $b \wedge c$ since P is prime. Furthermore, $e(b \wedge c) \subseteq e(b) \cap e(c)$, since any downset missing $b \wedge c$ must miss both the larger elements b and c . Thus, $P \in e(b \wedge c) \subseteq e(b) \cap e(c)$, as desired.

Finally, let G be a proper, nonempty, clopen downset of \mathcal{P} . Since G is open in $\mathcal{P} \setminus L$, G contains a basic open set around each of its points. Hence,

$$G = \bigcup_{P \in G} e(a_P),$$

where each a_P is chosen so that $P \in e(a_P) \subseteq G$. Since G is a closed subset of a compact space, G is also compact. Hence,

$$G = \bigcup_{i=1}^n e(a_i),$$

for some finite subcollection $\{a_i \mid i \in \mathbb{N}\}$ of $\{a_P \mid P \in G\}$. Finally, we claim that

$$\bigcup_{i=1}^n e(a_i) = e\left(\bigvee_{i=1}^n a_i\right),$$

thus finishing the proof. For ease of notation, we show the claim for $n = 2$ and reach the desired conclusion via induction. Let now $P \in e(b) \cup e(c)$. By

definition, P misses one of b or c . Since P is a downset, it must also miss the larger element $b \vee c$. Next, let $P \in e(b \vee c)$. Since P misses $b \vee c$, it must miss at least one of b or c , as P is join-closed. Hence, $P \in e(b) \cup e(c)$. \square