On Modularity and Distributivity in Lattices

Math 743 - Spring 2010

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Whenever an inequality is mentioned in the following two theorems, the reverse inequality holds in *all* lattices.

1 Modularity

Theorem 1.1. *The following are equivalent for a lattice* $\boldsymbol{L}$*:*

a. $\boldsymbol{L}$ *is modular.*

b. For all $x, y, z \in L$ *with* $x \geq z$, $x \wedge (y \vee z) \leq (x \wedge y) \vee z$.

c. For all $x, y, z \in L$ *with* $x \leq z$, $x \vee (y \wedge z) \geq (x \vee y) \wedge z$.

d. $(w \vee z) \wedge (y \vee z) \leq ((w \vee z) \wedge y) \vee z$ *for all* $w, y, z \in L$.

e. $(w \wedge z) \vee (y \wedge z) \geq ((w \wedge z) \vee y) \wedge z$ *for all* $w, y, z \in L$.

f. N_5 *is not isomorphic to a sublattice of* $\boldsymbol{L}$.

g. The transposition maps ϕ_a *and* ψ_b *are isomorphisms that invert each other for all* $a, b \in L$.

h. Whenever L *is finite,* $a \prec a \vee b \Rightarrow a \wedge b \prec b$ *and* $a \wedge b \prec a \Rightarrow b \prec a \vee b$ *for all* $a, b \in L$.

2 Distributivity

Theorem 2.1. *The following are equivalent for a lattice* $\boldsymbol{L}$*:*

a. $\boldsymbol{L}$ *is distributive.*

b. $x \wedge (y \vee z) \leq (x \wedge y) \vee (x \wedge z)$ *for all* $x, y, z \in L$*.*

c. $x \vee (y \wedge z) \geq (x \vee y) \wedge (x \vee z)$ *for all* $x, y, z \in L$*.*

d. $(x \vee y) \wedge (x \vee z) \wedge (y \vee z) = (x \wedge y) \vee (x \wedge z) \vee (y \wedge z)$ *for all* $x, y, z \in \boldsymbol{L}$*.*

e. Neither N_5 *nor* M_3 *is isomorphic to a sublattice of* $\boldsymbol{L}$*.*

f. $\boldsymbol{L}$ *is modular and* M_3 *is not isomorphic to a sublattice of* $\boldsymbol{L}$*.*

g. The transposition maps ϕ_a *and* ψ_a *are homomorphisms for all* $a \in L$*.*

h. Whenever L *is finite, the join irreducible and the join prime elements coincide.*