Math 731 Homework 2 Correction 1

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1 Problem 16B2

Proposition 1.1. *Any base for the open sets in a second countable space has a countable subfamily that is a base.*

Proof. Let X be a second countable space, $\mathcal{B} = \{B_\alpha \mid \alpha \in \Gamma\}$ be any base for X, and $\mathcal{C} = \{C_i \mid i \in \mathbb{N}\}$ be the countable base guaranteed by the second countability of X. Our aim is to show that, for each $i \in \mathbb{N}$, C_i can be represented as the union of countably-many members of $\mathcal{B}$ (call this countable subcollection $\mathcal{B}_{C_i}$). Since the union of countably-many sets each having countably-many members is again countable, the set $\{A \mid A \in \mathcal{B}_{C_i} \text{ for some } i\}$ will be a countable subfamily of $\mathcal{B}$ that is a base for X.

To finish the proof, let $C_k \in \mathcal{C}$. Since $\mathcal{B}$ is a base for X, we can write $C_k = \bigcup_{i \in I} B_i$ for some subset I of the indexing set Γ. Now, for each $x \in C_k$, pick a set $B_{i_x} \in \mathcal{B}$ with $x \in B_{i_x}$ and $i_x \in I$. Since $\mathcal{C}$ is also a base for X, we can find some C_{i_x} with $x \in C_{i_x} \subset B_{i_x}$. It follows that $\{B_{i_x} \mid x \in C_k\}$ is a countable set (we chose one element of $\mathcal{B}$ for each element of the countable set $\mathcal{C}$) whose union is C_i (by construction, every element of C_i is present in the union and every B_{i_x} is contained in C_i). □

2 Problem 16B3

Proposition 2.1. *Any increasing chain of real numbers that is well ordered by the usual order must be countable.*

Proof. Let A be a set of real numbers that is well-ordered by the usual order. We claim that, for each $a \in A$, there is $n_a \in \mathbb{N}$ such that $\left(a, a + \frac{1}{n_a}\right] \cap A = \emptyset$.

Observe that the truth of this claim implies the countability of A, as each interval will contain a distinct rational number. Suppose now, for the purpose of contradiction, that the claim does not hold. That is, there exists some $b \in A$ such that, for all $n \in \mathbb{N}$, $\left(b, b + \frac{1}{n}\right] \cap A \neq \emptyset$. It follows that the set $A \setminus \{c \in A \mid c \leq b\}$ has no least element, which is a contradiction with the fact that A is well-ordered, thus completing the proof. $\square$