Math 731 Homework 1 (Correction 1)

Austin Mohr

January 25, 2010

1 Problem 15B1

Definition 1.1. *A space X is said to be completely normal if every subspace of X is normal.*

Proposition 1.2. *X is completely normal if and only if, whenever A and B are subsets of X with $A \cap \mathrm{Cl}_X(B) = \mathrm{Cl}_X(A) \cap B = \emptyset$, then there are disjoint open sets $U \supset A$ and $V \supset B$.*

Proof. ($\Rightarrow$) Let A and B be given with $A \cap \mathrm{Cl}_X(B) = \mathrm{Cl}_X(A) \cap B = \emptyset$. Denote by Y the subspace $X - (\mathrm{Cl}_X(A) \cap \mathrm{Cl}_X(B))$. Now, both A and B are contained in Y. To see this, observe that

$$\begin{aligned} A \cap (\mathrm{Cl}_X(A) \cap \mathrm{Cl}_X(B)) &= (A \cap \mathrm{Cl}_X(B)) \cap \mathrm{Cl}_X(A) \\ &= \emptyset \cap \mathrm{Cl}_X(A) \\ &= \emptyset. \end{aligned}$$

Similarly,

$$\begin{aligned} B \cap (\mathrm{Cl}_X(A) \cap \mathrm{Cl}_X(B)) &= (\mathrm{Cl}_X(A) \cap B) \cap \mathrm{Cl}_X(B) \\ &= \emptyset \cap \mathrm{Cl}_X(B) \\ &= \emptyset. \end{aligned}$$

Next, we claim that $\mathrm{Cl}_Y(A)$ and $\mathrm{Cl}_Y(B)$ are disjoint, closed sets in Y. For disjointness, obvsere that

$$\begin{aligned} \mathrm{Cl}_Y(A) \cap \mathrm{Cl}_Y(B) \cap Y &\subset \mathrm{Cl}_X(A) \cap \mathrm{Cl}_X(B) \cap Y \\ &= \emptyset. \end{aligned}$$

For closedness, note that the closure of a set in a topological space is always closed in that space.

By the complete normality of X, we have that Y is normal. Hence, there exist sets U and V that are open in Y (and so open in X) with $U \supset \mathrm{Cl}_Y(A)$ and $V \supset \mathrm{Cl}_Y(B)$. As A and B are both contained in Y, we further have that $\mathrm{Cl}_Y(A) \supset A$ and $\mathrm{Cl}_Y(B) \supset B$. Therefore, $U \supset A$ and $V \supset B$, as desired.

($\Leftarrow$) Let Y be a subspace of X and let A and B be disjoint closed subsets of Y. Observe that

$$\begin{aligned} A \cap \mathrm{Cl}_X(B) &= A \cap B \\ &= \emptyset, \end{aligned}$$

and similarly

$$\begin{aligned} \mathrm{Cl}_X(A) \cap B &= A \cap B \\ &= \emptyset. \end{aligned}$$

By hypothesis, there are disjoint open subsets U and V of X with $U \supset A$ and $V \supset B$. It follows that the sets $U \cap Y$ and $V \cap Y$ are disjoint open subsets of Y with $(U \cap Y) \supset A$ and $(V \cap Y) \supset B$. Hence, Y is normal, and so X is completely normal. □

2 Problem 15B3

Lemma 2.1. *If M is metrizable and $N \subset M$, then the subspace N is metrizable with the topology generated by the restriction of any metric which generates the topology on M.*

Proof. Let τ be the topology on M generated by a metric ρ. Let σ be the relative topology on N and let ρ_N be the restriction of ρ to N. We show that σ is generated by ρ_N.

Let $O \in \sigma$. It must be that $O = N \cap G$ for some $G \in \tau$. Since M is generated by ρ, we know that $G = \bigcup_{x \in G} B_\rho(x, \epsilon_x)$, where $\epsilon_x > 0$ may depend on x. Now,

$$\begin{aligned} O &= N \cap G \\ &= N \cap \bigcup_{x \in G} B_\rho(x, \epsilon_x) \\ &= \bigcup_{x \in G} N \cap B_\rho(x, \epsilon_x) \\ &= \bigcup_{x \in N \cap G} B_{\rho_N}(x, \epsilon_x). \end{aligned}$$

Hence, O is the union of open balls with respect to the metric ρ_N. Therefore, σ is generated by the ρ_N, as desired. □

Proposition 2.2. *Every metric space is completely normal.*

Proof. By the lemma, any subspace of a metric space is itself a metric space. As every metric space is T_4 (hence, normal), it follows that every subspace of a metric space is normal. That is, every metric space is completely normal. □