Math 730 Homework 9

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1 Problem 6B3

Proposition 1.1. *An open subset of a separable space is separable.*

Proof. Let O be an open subset of the separable space X, and let D be the countable, dense set in X. Consider any open neighborhood G of a point $x \in O$ with the constraint that G be contained in O (since O is open, this can always be done). Since D is dense in X, $G \cap D \neq \emptyset$. Hence, any open neighborhood of a point in O contains an element of D. In other words, D is dense in O. □

2 Problem 7B

Recall that the Cantor-Bernstein theorem states that if A and B are sets and if one-to-one functions $f : A \to B$ and $g : B \to A$ exist, then a one-to-one function of A onto B exists. The analog for topological spaces would be as follows: Whenever X can be embedded in Y and Y can be embedded in X, then X and Y are homeomorphic.

Proposition 2.1. *The aforementioned analog of the Cantor-Bernstein theorem for topological spaces is false.*

Proof. Define $\mathbb{R}^{\geq 1}$ to be the set $\{x \in \mathbb{R} \mid x \geq 1\}$.

Claim 1. *The function*

$$f : [0,1] \to \mathbb{R}^{\geq 1}$$
$$f(x) = x + 1$$

is an embedding of $[0,1]$ *into* $\mathbb{R}^{\geq 1}$.

Proof. The function f is one-to-one and continuous. The inverse of f, given by $f^{-1}(x) = x - 1$, is also continuous. □

Claim 2. *The function*

$$g : \mathbb{R}^{\geq 1} \to [0,1]$$
$$g(x) = \frac{1}{x}$$

is an embedding of $\mathbb{R}^{\geq 1}$ *into* $[0,1]$.

Proof. The function g is one-to-one and continuous. The inverse of g, given by $g^{-1}(x) = x$, is also continuous. □

Now, the property that every continuous, real-valued function on some set achieves its maximum is a topology property. As $[0,1]$ is a compact set, it possesses this property. The property does not hold, however, for $\mathbb{R}^{\geq 1}$ (the identity function on $\mathbb{R}^{\geq 1}$ serves as a counterexample). Therefore, while there exists an embedding of $[0,1]$ into $\mathbb{R}^{\geq 1}$ and vice versa, the two spaces are not homeomorphic, and so the proposed analog of the Cantor-Bernstein theorem is not true in general. □

3 Problem 8D

Let X and Y be topological spaces containing subsets A and B, respectively.

Proposition 3.1. *In the product space* $X \times Y$, $(A \times B)^\circ = A^\circ \times B^\circ$.

Proof. It follows directly that

$$\begin{aligned}
(x,y) \in (A \times B)^\circ &\Leftrightarrow (x,y) \in G \text{ for some open set } G \subset A \times B \\
&\Leftrightarrow (x,y) \in G_1 \times G_2 \text{ for some open sets } G_1 \subset A \text{ and } G_2 \subset B \\
&\Leftrightarrow x \in G_1 \text{ and } y \in G_2 \text{ for some open sets } G_1 \subset A \text{ and } G_2 \subset B \\
&\Leftrightarrow x \in A^\circ \text{ and } y \in B^\circ \\
&\Leftrightarrow (x,y) \in A^\circ \times B^\circ.
\end{aligned}$$

□

Proposition 3.2. *In the product space $X \times Y$, $\overline{A \times B} = \overline{A} \times \overline{B}$.*

Proof. To see that $\overline{A \times B} \subset \overline{A} \times \overline{B}$, observe that

$$\begin{aligned}
(x, y) \in \overline{A \times B} &\Rightarrow (x, y) \in F \text{ for all closed sets } F \supset A \times B \\
&\Rightarrow (x, y) \in F_1 \times F_2 \text{ for all closed sets } F_1 \supset A \text{ and } F_2 \supset B \\
&\Rightarrow x \in F_1 \text{ and } y \in F_2 \text{ for all closed sets } F_1 \supset A \text{ and } F_2 \supset B \\
&\Rightarrow x \in \overline{A} \text{ and } y \in \overline{B} \\
&\Rightarrow (x, y) \in \overline{A} \times \overline{B}.
\end{aligned}$$

To see that $\overline{A} \times \overline{B} \subset \overline{A \times B}$, observe that

$$\begin{aligned}
(x, y) \in \overline{A} \times \overline{B} &\Rightarrow x \in \overline{A} \text{ and } y \in \overline{B} \\
&\Rightarrow \text{ for all basic open } O_x \text{ containing } x,\ O_x \cap A \neq \emptyset, \\
&\qquad \text{and for all basic open } O_y \text{ containing } y,\ O_y \cap B \neq \emptyset \\
&\Rightarrow \text{ for all basic open } O \text{ containing } (x, y),\ O \cap (A \times B) \neq \emptyset \\
&\Rightarrow (x, y) \in \overline{A \times B}.
\end{aligned}$$

□