Math 730 Homework 8 (Correction 1)

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1 Extra Problem

Proposition 1.1. *Let X and Y be topological spaces and $f : X \to Y$ a bijection. The following are equivalent:*

a.) The function f is a homeomorphism.

b.) For any $G \subset X$, $f^{\to}G$ is open in Y if and only if G is open in X.

c.) For any $F \subset X$, $f^{\to}F$ is closed in Y if and only if F is closed in X.

d.) For any $E \subset X$, $f^{\to}\overline{E} = \overline{f^{\to}E}$.

Proof. (We have shown $a \Leftrightarrow b \Leftrightarrow c$ in class.)

((a and b) $\Rightarrow d$) Let f be a homeomorphism (and so also possesses the property that, for any $G \subset X$, $f^{\to}G$ is open in Y if and only if G is open in X).

We show first that $f^{\to}\overline{E} \subset \overline{f^{\to}E}$. To that end, let $b \in f^{\to}\overline{E}$ and consider any open O containing b. By the continuity of f, $f^{\leftarrow}O$ is open. Furthermore, there is an element a of $f^{\leftarrow}O$ such that $f(a) = b$. Now, since $a \in \overline{E}$, any open set containing a intersects nontrivially with E. In particular, the open set $f^{\leftarrow}O$ intersects nontrivially with E, and so $f^{\to}(f^{\leftarrow}O)$ intersects nontrivially with $f^{\to}E$. As f is a bijection, $f^{\to}(f^{\leftarrow}O) = O$, and so we see that O intersects nontrivially with $f^{\to}E$. In other words, $b \in \overline{f^{\to}E}$, as desired.

Next we show that $\overline{f^{\to}E} \subset f^{\to}\overline{E}$. To that end, let $b \in \overline{f^{\to}E}$ and suppose, for the purpose of contradiction, that $b \notin f^{\to}\overline{E}$. In other words, $b = f(a)$ for $a \notin \overline{E}$ (such an a must exist, since f is a bijection). Hence, we can find an open ball O containing a such that O intersects trivially with E, which implies that $f^{\to}O$ is an open set containing b that intersects trivially with $f^{\to}E$. In other words, $b \notin \overline{f^{\to}E}$, which is a contradiction. Therefore, $\overline{f^{\to}E} \subset f^{\to}\overline{E}$.

($d \Rightarrow c$) Let f be such that, for any $E \subset X$, $f^{\to}\overline{E} = \overline{f^{\to}E}$.

We show first that, for any $F \subset X$, $f^{\to}F$ is closed in Y implies that F is closed in X.

For that, observe

$$\begin{aligned} f^{\to}F \text{ is closed in } Y &\Rightarrow f^{\to}F = \overline{f^{\to}F} \\ &\Rightarrow f^{\to}F = f^{\to}\overline{F} \\ &\Rightarrow F = \overline{F} && \text{(since } f \text{ is a bijection)} \\ &\Rightarrow F \text{ is closed in } X. \end{aligned}$$

We show next that, for any $F \subset X$, F is closed in X implies that $f^{\to}F$ is closed in Y. For that, observe

$$\begin{aligned} F \text{ is closed in } X &\Rightarrow F = \overline{F} \\ &\Rightarrow f^{\to}F = f^{\to}\overline{F} \\ &\Rightarrow f^{\to}F = \overline{f^{\to}F} \\ &\Rightarrow f^{\to}F \text{ is closed in } Y. \end{aligned}$$

Therefore, for any $F \subset X$, $f^{\to}F$ is closed in Y if and only if F is closed in X. □

2 Problem 7A

Definition 2.1. *The characteristic function of a subset A of a set X (denoted χ_A) is the function from X to $\mathbb{R}$ which is 1 at points of A and 0 at other points of X.*

Proposition 2.2. *The characteristic function of A is continuous if and only if A is both open and closed in X.*

Proof. ($\Rightarrow$) Let χ_A be continuous. Observe $\chi_A : X \to \{0,1\}$ is indeed a function between topological spaces when each subset of $\mathbb{R}$ is taken together with its relative topology. Now,

$$f^{\leftarrow}(\{1\}) = A.$$

As $\{1\}$ is closed in $\{0,1\}$, A is closed in X (by the continuity of χ_A). Similarly, we have

$$f^{\leftarrow}(\{0\}) = A^c.$$

As $\{0\}$ is closed in $\{0,1\}$, A^c is closed in X (by the continuity of χ_A), and so A is open in X.

($\Leftarrow$) Let A be both open and closed in X and consider an arbitrary open subset O of $\mathbb{R}$. We show that $f^{\leftarrow}(B \cap O)$ is open for all subsets B of $\{0,1\}$.

$$\begin{aligned} f^{\leftarrow}(\emptyset) &= \emptyset, \text{ which is open} \\ f^{\leftarrow}(\{0\}) &= A^c, \text{ which is open, since } A \text{ is closed} \\ f^{\leftarrow}(\{1\}) &= A, \text{ which is open} \\ f^{\leftarrow}(\{0,1\}) &= X, \text{ which is open} \end{aligned}$$

Therefore, χ_A is continuous. □

Proposition 2.3. *The topological space X has the discrete topology if and only if $f : X \to Y$ is continuous whenever (Y, τ) is a topological space.*

Proof. ($\Rightarrow$) Let X have the discrete topology. That is, every subset A of X is open. In particular, $f^{\leftarrow}(O)$ is open in X for any open subset O of Y. Therefore, f is continuous.

($\Leftarrow$) Let $f : X \to Y$ be continuous whenever (Y, τ) is a topological space. In particular, let $Y = X$, τ be the discrete topology, and f be the identity on X. Now, for any subset A of the codomain, we have that A is open (since the codomain has the discrete topology). Hence, by the continuity of f, $f^{\leftarrow}(A)$ is open. At the same time, by the definition of f, $f^{\leftarrow}(A) = A$. Therefore, every subset A of the domain is open, and so X has the discrete topology. □

Proposition 2.4. *The topological space X has the trivial topology if and only if $f : Y \to X$ is continuous whenever (Y, τ) is a topological space.*

Proof. ($\Rightarrow$) Let X have the trivial topology. To establish the continuity of f, we show that the preimage of any open set in X is open in Y. As X has the trivial topology, it suffices to observe that

$$
\begin{aligned}
f^{\leftarrow}(\emptyset) &= \emptyset \text{ which is open in } Y \\
f^{\leftarrow}(X) &= Y \text{ which is open in } Y.
\end{aligned}
$$

Therefore, f is continuous.

($\Leftarrow$) Let $f : Y \to X$ be continuous whenever (Y, τ) is a topological space. In particular, let $Y = X$, τ be the trivial topology, and f be the identity on X. Now, consider any open set A in X. Since f is continuous, $f^{\leftarrow}(A)$ is open in Y. As the domain has the trivial topology, it must be that $f^{\leftarrow}(A)$ is either the empty set or all of X. At the same time, since f is the identity on X, we see that A is either the empty set or all of X. Therefore, the only open sets in the codomain are the empty set or X, and so the codomain has the trivial topology. □