

Math 730 Homework 5 (Correction 1)

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1 Extra Problem 1

Lemma 1.1. *Let $\hat{\cdot}$ be a Čech closure operation. If $A \subset B$, then $\hat{A} \subset \hat{B}$.*

Proof. It follows directly from the properties of Čech closure operations that

$$\begin{aligned} A \subset B &\Rightarrow A \cup B = B \\ &\Rightarrow \widehat{A \cup B} = \widehat{B} \\ &\Rightarrow \hat{A} \cup \hat{B} = \hat{B}. \end{aligned}$$

Now, $\hat{A} \subset \hat{A} \cup \hat{B} = \hat{B}$, and so $\hat{A} \subset \hat{B}$, as desired. \square

Proposition 1.2. *If we define a set in a Čech closure space $(X, \hat{\cdot})$ to be closed if $A = \hat{A}$, then the result is a topology.*

Proof. Let \mathcal{F} denote the collection of subsets of X that are closed with respect to $\hat{\cdot}$. By Homework 4, Problem 2, if \mathcal{F} satisfies

(F-a) the intersection of an arbitrary collection of elements of \mathcal{F} belongs to \mathcal{F} ,

(F-b) the union of a finite collection of elements of \mathcal{F} belongs to \mathcal{F} , and

(F-c) the sets \emptyset and X belong to \mathcal{F} ,

then $\tau = \{F^c \mid F \in \mathcal{F}\}$ is a topology on X . We verify that \mathcal{F} indeed satisfies each of these three properties.

Claim. *The intersection of an arbitrary collection of elements of \mathcal{F} belongs to \mathcal{F} .*

Proof. Let $F_\alpha \in \mathcal{F}$ for all α belonging to some indexing set I . As $\hat{\cdot}$ is a Čech closure operation,

$$\bigcap_{\alpha \in I} F_\alpha \subset \widehat{\bigcap_{\alpha \in I} F_\alpha}.$$

To see the reverse inclusion, observe that

$$\begin{aligned} &\bigcap_{\alpha \in I} F_\alpha \subset F_\alpha \text{ for each } \alpha \in I \\ \Rightarrow &\widehat{\bigcap_{\alpha \in I} F_\alpha} \subset \widehat{F_\alpha} \text{ for each } \alpha \in I \quad (\text{by 1.1}) \\ \Rightarrow &\widehat{\bigcap_{\alpha \in I} F_\alpha} \subset \bigcap_{\alpha \in I} \widehat{F_\alpha} \\ \Rightarrow &\widehat{\bigcap_{\alpha \in I} F_\alpha} \subset \bigcap_{\alpha \in I} F_\alpha \quad (\text{as each } F_\alpha \text{ is closed}). \end{aligned}$$

Hence, $\bigcap_{\alpha \in I} F_\alpha = \widehat{\bigcap_{\alpha \in I} F_\alpha}$ (i.e. $\bigcap_{\alpha \in I} F_\alpha$ is closed in \mathcal{F}). \square

Claim. *The union of a finite collection of elements of \mathcal{F} belongs to \mathcal{F} .*

Proof. Let $F_i \in \mathcal{F}$ for $1 \leq i \leq n$. We have that

$$\begin{aligned} \bigcup_{i=1}^n F_i &= \bigcup_{i=1}^n \hat{F}_i && \text{(since } F_i = \hat{F}_i \text{ for all } i) \\ &= \widehat{\bigcup_{i=1}^n F_i} && \text{(by induction on } i, \text{ using the fact that } \widehat{A \cup B} = \hat{A} \cup \hat{B} \text{ for the Čech closure operation } \hat{}) \end{aligned}$$

□

Claim. *The sets \emptyset and X belong to \mathcal{F} .*

Proof. We have that $\emptyset = \hat{\emptyset}$ by definition of Čech closure operation, so $\emptyset \in \mathcal{F}$. Now,

$$\begin{aligned} X &\subset \hat{X} && \text{(since } \hat{} \text{ is a Čech closure operation)} \\ &\subset X && \text{(since } \hat{} : \mathcal{P}(X) \rightarrow \mathcal{P}(X)). \end{aligned}$$

Hence, $X = \hat{X}$, and so $X \in \mathcal{F}$.

□

Therefore, $\tau = \{F^c \mid F \in \mathcal{F}\}$ is a topology on X , as desired.

□

2 Problem 3A1

Proposition 2.1. *If \mathcal{F} is the collection of all closed, bounded subsets of \mathbb{R} (in its usual topology), together with \mathbb{R} itself, then \mathcal{F} is the family of closed sets for a topology on \mathbb{R} strictly weaker than the usual topology.*

Proof. We proceed by verifying properties F-a, F-b, and F-c on the family of sets \mathcal{F} .

Claim. *The intersection of an arbitrary collection of elements of \mathcal{F} belongs to \mathcal{F} .*

Proof. Let $F_\alpha \in \mathcal{F}$ for all α belonging to some indexing set I . In \mathbb{R} , the intersection of an arbitrary collection of closed sets is closed. Hence, $\bigcap_{\alpha \in I} F_\alpha$ is closed. Furthermore, each F_α is bounded, and so $\bigcap_{\alpha \in I} F_\alpha$ (which is contained in F_α for every α) is also bounded. Thus, $\bigcap_{\alpha \in I} F_\alpha$ is closed and bounded, and so belongs to \mathcal{F} . □

Claim. *The union of a finite collection of elements of \mathcal{F} belongs to \mathcal{F} .*

Proof. Let $F_i \in \mathcal{F}$ for all $1 \leq i \leq n$. In \mathbb{R} , the union of a finite collection of closed sets is closed. Hence, $\bigcup_{i=1}^n F_i$ is closed. Now, each F_i is bounded by some $B(0, \epsilon_i)$. Take $\epsilon = \max\{\epsilon_i \mid 1 \leq i \leq n\}$. We see that $F_i \subset B(0, \epsilon)$ for all $1 \leq i \leq n$, and so $\bigcup_{i=1}^n F_i \subset B(0, \epsilon)$. Thus, $\bigcup_{i=1}^n F_i$ is closed and bounded, and so belongs to \mathcal{F} . □

Claim. *The sets \emptyset and \mathbb{R} belong to \mathcal{F} .*

Proof. We have that $\mathbb{R} \in \mathcal{F}$ by definition. Now, \emptyset is trivially closed and $\emptyset \subset B(0, 1)$. Hence, \emptyset is closed and bounded, and so belongs to \mathcal{F} . □

Therefore, \mathcal{F} is the family of closed subsets for the topology $\tau = \{F^c \mid F \in \mathcal{F}\}$. To see that τ is weaker than the usual Euclidean topology (call it τ'), observe that $B(0, 1) \in \tau'$ (since open balls are open), but $B(0, 1) \notin \tau$ (since $B(0, 1)^c = \mathbb{R} \setminus B(0, 1)$ is unbounded). □

3 Problem 3E2

Proposition 3.1. *Let X be a metrizable space whose topology is generated by a metric ρ . The closure of a set $E \subset X$ is given by*

$$\bar{E} = \{y \in X \mid \rho(E, y) = 0\}.$$

Proof. (⊂) Let $x \in \bar{E}$. It follows that,

$$\begin{aligned} x \in \bar{E} &\Rightarrow G \cap E \neq \emptyset \text{ for all open } G \text{ containing } x \\ &\Rightarrow B(x, \epsilon) \cap E \neq \emptyset \text{ for all } \epsilon > 0 \\ &\Rightarrow d(E, x) < \epsilon \text{ for all } \epsilon > 0 \\ &\Rightarrow d(E, x) = 0 \\ &\Rightarrow x \in \{y \in X \mid \rho(E, y) = 0\}. \end{aligned}$$

Hence, $\bar{E} \subset \{y \in X \mid \rho(E, y) = 0\}$.

(⊃) Let $x \in \{y \in X \mid \rho(E, y) = 0\}$. It follows that,

$$\begin{aligned} x \in \{y \in X \mid \rho(E, y) = 0\} &\Rightarrow \rho(E, x) = 0 \\ &\Rightarrow B(x, \epsilon) \cap E \neq \emptyset \text{ for all } \epsilon > 0. \end{aligned}$$

Now, for any open G containing x , there must exist $\epsilon_0 > 0$ such that $B(x, \epsilon_0) \subset G$ (by definition of openness in a metrizable space). Hence,

$$\begin{aligned} B(x, \epsilon) \cap E \neq \emptyset \text{ for all } \epsilon > 0 &\Rightarrow G \cap E \neq \emptyset \text{ for all open } G \text{ containing } x \\ &\Rightarrow x \in \bar{E}. \end{aligned}$$

Hence, $\{y \in X \mid \rho(E, y) = 0\} \subset \bar{E}$. □

4 Problem 3E3

Proposition 4.1. *Let X be a metrizable space whose topology is generated by a metric ρ . The closed disk $U(x, \bar{\epsilon}) = \{y \in X \mid \rho(x, y) \leq \bar{\epsilon}\}$ is closed in X , but may not be the closure of the open disk $U(x, \epsilon)$.*

Proof. We show, equivalently, that $U(x, \bar{\epsilon})^c$ is open. As X is a metrizable space, we need to establish that, for any $z \in U(x, \bar{\epsilon})^c$, there exists $\epsilon' > 0$ such that $B(z, \epsilon') \subset U(x, \bar{\epsilon})^c$. To that end, let $\epsilon' = \rho(x, z) - \epsilon$ (which, a priori, may or may not be positive). Now,

$$\begin{aligned} B(z, \epsilon') \subset U(x, \bar{\epsilon})^c &\Leftrightarrow B(z, \epsilon') \subset \{y \in X \mid \rho(x, y) > \epsilon\} \\ &\Leftrightarrow \rho(x, w) > \epsilon \text{ for all } w \in B(z, \epsilon'). \end{aligned}$$

Applying the triangle inequality, we have

$$\begin{aligned} \rho(x, z) &\leq \rho(x, w) + \rho(w, z) \\ &< \rho(x, w) + (\rho(x, z) - \epsilon) \quad (\text{note now that } \rho(x, z) > \epsilon, \text{ so } \rho(x, z) - \epsilon > 0). \end{aligned}$$

Rearranging the terms yields that $\rho(x, w) > \epsilon$, as desired.

As an example when $U(x, \bar{\epsilon}) \neq \bar{U}(x, \bar{\epsilon})$, let $X = \mathbb{R}$ and ρ be the discrete metric. Observe that $U(0, \bar{1}) = \mathbb{R}$, but

$$\begin{aligned} \overline{U(0, \bar{1})} &= \bigcap \{K \mid K \text{ is closed and contains } U(0, \bar{1})\} \\ &= \bigcap \{K \mid K \text{ is closed and contains } \{0\}\} \\ &= \{0\} \quad (\text{since } \{0\} \text{ itself is closed in a metric space}). \end{aligned}$$

□