Math 730 Homework 4 (Correction 1)

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1 Extra Problem 1

Proposition 1.1. *A set in a pseudometric space is open if and only if it is a union of open disks.*

Proof. ($\Rightarrow$) Let O be an open set in a pseudometric space. To be open means that, for each $x \in O$, there is $\epsilon_x > 0$ such that $B(x, \epsilon_x) \subset O$. We claim that

$$O = \bigcup_{x \in O} B(x, \epsilon_x).$$

To see this, choose $x_0 \in O$. By definition of openness, $x_0 \in B(x_0, \epsilon_{x_0})$, but $B(x_0, \epsilon_{x_0}) \subset \bigcup_{x \in O} B(x, \epsilon_x)$, so $x_0 \in \bigcup_{x \in O} B(x, \epsilon_x)$.

To get the reverse inclusion, choose $x_0 \in \bigcup_{x \in O} B(x, \epsilon_x)$. By construction, $x_0 \in B(x_0, \epsilon_{x_0})$. Now, ϵ_{x_0} was chosen specifically to ensure that $B(x_0, \epsilon_{x_0}) \subset O$. Hence, $x \in O$.

Therefore, $O = \bigcup_{x \in O} B(x, \epsilon_x)$. That is, O is the union of open disks.

($\Leftarrow$) Let $\bigcup_{\alpha \in I} B_\alpha$ be an arbitrary union of open disks.

$$x \in \bigcup_{\alpha \in I} B_\alpha \quad \Rightarrow \quad x \in B_{\alpha_0} \text{ for some } \alpha_0 \in I$$

As open disks are open, we have successfully found an open set contained in $\bigcup_{\alpha \in I} B_\alpha$ that contains x. Therefore, $\bigcup_{\alpha \in I} B_\alpha$ is open. □

2 Extra Problem 2

Proposition 2.1. *Let $\mathcal{F}$ be any collection of subsets of a set X. If $\mathcal{F}$ satisfies*

(i) $\bigcap_{\alpha \in I} F_\alpha \in \mathcal{F}$ *if* $F_\alpha \in \mathcal{F}$ *for all* α *belonging to some indexing set* I

(ii) $\bigcup_{i=1}^{n} F_i \in \mathcal{F}$ *if* $F_i \in \mathcal{F}$ *for all* i

(iii) $\emptyset \in \mathcal{F}$ *and* $X \in \mathcal{F}$,

then $\tau = \{F^c \mid F \in \mathcal{F}\}$ *is a topology on* X *and* $\mathcal{F}$ *is the collection of closed sets in this topology.*

Proof. We show that τ satisfies each of the three properties of topologies on X.

Claim. $\bigcup_{\alpha \in I} F_\alpha^c \in \tau$ *if* $F_\alpha^c \in \tau$ *for all* α *belonging to some indexing set* I.

Proof. By problem 1B1, we have

$$\bigcup_{\alpha \in I} F_\alpha^c = \left(\bigcap_{\alpha \in I} F_\alpha \right)^c .$$

By hypothesis, $\bigcap_{\alpha \in I} F_\alpha$ is closed, and so its complement is open. Hence, $\bigcup_{\alpha \in I} F_\alpha^c \in \tau$. ☐

Claim. $\bigcap_{i=1}^{n} F_i^c \in \tau$ *if* $F_i^c \in \tau$ *for all* i.

Proof. As a corollary of problem 1B1 (take the complement of both sides), we have

$$\bigcap_{i=1}^{n} F_i^c = \left(\bigcup_{i=1}^{n} F_i \right)^c .$$

By hypothesis, $\bigcup_{i=1}^{n} F_i$ is closed, and so its complement is open. Hence, $\bigcap_{i=1}^{n} F_i^c \in \tau$. ☐

Claim. $\emptyset \in \tau$ *and* $X \in \tau$.

Proof. We have that $\emptyset = X^c$ and $X \in \mathcal{F}$, so $\emptyset \in \tau$. Also, $X = \emptyset^c$ and $\emptyset \in \mathcal{F}$, so $X \in \tau$. ☐

Therefore, τ is a topology on X.

It remains to show that $\mathcal{F}$ is the collection of closed sets in τ. We have immediately that any element $F \in \mathcal{F}$ is closed (since $F^c \in \tau$, and so open). Similarly, if a subset K of X is closed, then K^c is open. Thus, $K^c \in \tau$, which implies that $K \in \mathcal{F}$ (τ is precisely the collection of complements of sets from $\mathcal{F}$). Therefore, the collection of closed sets in τ and the collection $\mathcal{F}$ conincide. ☐

3 Problem 2F2

Definition 3.1. *Let* ρ *be a bounded metric on* M*; that is, for some constant* A,

$$\rho(x, y) \leq A \text{ for all } x, y \in M.$$

Let $\mathcal{F}(M)$ *be all nonempty closed subsets of* M *and for* $A, B \in \mathcal{F}(M)$ *define*

$$d_A(B) = \sup\{\rho(A, x) \mid x \in B\}$$
$$d(A, B) = \max\{d_A(B), d_B(A)\}.$$

Lemma 3.2. *For all* $A, B, C \in \mathcal{F}(M)$,

$$\sup\{\rho(A, c) \mid c \in C\} \leq \sup\{\rho(A, b) + \rho(B, c) \mid b \in B, c \in C\}.$$

Proof. For all $A, B, C \in \mathcal{F}(M)$,

$$\begin{aligned} \sup\{\rho(A, c) \mid c \in C\} &= \sup\{\inf\{\rho(a, c) \mid a \in A\} \mid c \in C\} \\ &\leq \sup\{\inf\{\rho(a, b) + \rho(B, c) \mid a \in A, b \in B\} \mid c \in C\} \\ &= \sup\{\inf\{\rho(a, b) \mid a \in A, b \in B\} + \rho(B, c) \mid c \in C\} \\ &\leq \sup\{\inf\{\rho(a, b) \mid a \in A\} + \rho(B, c) \mid b \in B, c \in C\} \\ &= \sup\{\rho(A, b) + \rho(B, c) \mid b \in B, c \in C\}. \end{aligned}$$

☐

Proposition 3.3. *The function d is a metric on $\mathcal{F}(M)$ with the property that $d(\{x\},\{y\}) = \rho(x,y)$ (called the Hausdorff metric on $\mathcal{F}(M)$).*

Proof. We show that d satisfies each of the three properties of metrics on $\mathcal{F}(M)$.

Claim. *For all $A, B \in \mathcal{F}(M)$, $d(A,B) \geq 0$ with equality if and only if $A = B$.*

Proof. As ρ is a metric on M,

$$\begin{aligned} \rho(x,y) \geq 0 \text{ for all } x,y \in M \quad &\Rightarrow \quad d_A(B) \geq 0 \text{ for all } A,B \in \mathcal{F}(M) \\ &\Rightarrow \quad d(A,B) \geq 0 \text{ for all } A,B \in \mathcal{F}(M). \end{aligned}$$

Now,

$$\begin{aligned} d(A,B) = 0 \quad &\Leftrightarrow \quad d_A(B) = 0 \text{ and } d_B(A) = 0 \\ &\Leftrightarrow \quad \sup\{\rho(A,b) \mid b \in B\} = 0 \text{ and } \sup\{\rho(B,a) \mid a \in A\} = 0 \\ &\Leftrightarrow \quad \rho(A,b) = 0 \text{ for all } b \in B \text{ and } \rho(B,a) = 0 \text{ for all } a \in A \\ &\Leftrightarrow \quad A \subset B \text{ and } B \subset A \qquad \text{(since } A \text{ and } B \text{ are closed)} \\ &\Leftrightarrow \quad A = B. \end{aligned}$$

□

Claim. *For all $A, B \in \mathcal{F}(M)$, $d(A,B) = d(B,A)$.*

Proof. For all $A, B \in \mathcal{F}(M)$,

$$\begin{aligned} d(A,B) &= \max\{d_A(B), d_B(A)\} \\ &= \max\{d_B(A), d_A(B)\} \\ &= d(B,A). \end{aligned}$$

□

Claim. *For all $A, B, C \in \mathcal{F}(M)$, $d(A,C) \leq d(A,B) + d(B,C)$.*

Proof. Let $A, B, C \in \mathcal{F}(M)$. Without loss of generality, assume A and C are such that

$$\begin{aligned} d(A,C) &= \max\{d_A(C), d_C(A)\} \\ &= d_A(C). \end{aligned}$$

Now,

$$\begin{aligned} d(A,C) &= d_A(C) \\ &= \sup\{\rho(A,c) \mid c \in C\} \\ &\leq \sup\{\rho(A,b) + \rho(B,c) \mid b \in B, c \in C\} \qquad \text{(by 3.2)} \\ &= \sup\{\rho(A,b) \mid b \in B\} + \sup\{\rho(B,c) \mid c \in C\} \\ &= d_A(B) + d_B(C) \\ &\leq \max\{d_A(B),\ d_B(A)\} + \max\{d_B(C),\ d_C(B)\} \\ &= d(A,B) + d(B,C). \end{aligned}$$

□

Therefore, d is a metric on $\mathcal{F}(M)$.

Moreover, for all $x, y \in M$,

$$\begin{aligned} d(\{x\},\{y\}) &= \max\{d_{\{x\}}(\{y\}), d_{\{y\}}(\{x\})\} \\ &= \max\{\sup\{\rho(\{x\},z) \mid z \in \{y\}\}, \sup\{\rho(\{y\},z) \mid z \in \{x\}\}\} \\ &= \max\{\sup\{\rho(x,y)\}, \sup\{\rho(y,x)\}\} \\ &= \max\{\rho(x,y), \rho(y,x)\} \\ &= \max\{\rho(x,y), \rho(x,y)\} \\ &= \rho(x,y). \end{aligned}$$

□