Austin Mohr
Math 730
Homework 3

Proposition. *Let (M, ρ) and (N, σ) be pseudometric spaces. If $f : M \to N$ is an isometry, then it is continuous.*

Proof. We aim to establish an alternate phrasing of the notion of continuity, namely that f is continuous at $x \in M$ if, for all $\epsilon > 0$, there exists $\delta > 0$ such that

$$f^{\rightarrow} B_\rho(x, \delta) \subset B_\sigma(f(x), \epsilon).$$

Given $\epsilon > 0$, take $\delta = \epsilon$ and let $x_0 \in B_\rho(x, \epsilon)$. Observe first that $f(x_0) \in f^{\rightarrow} B_\rho(x, \epsilon)$ by definition. Now,

$$\begin{aligned} x_0 \in B_\rho(x, \epsilon) &\Rightarrow \rho(x, x_0) < \epsilon \\ &\Rightarrow \sigma(f(x), f(x_0)) < \epsilon \qquad (f \text{ is an isometry, so } \sigma(f(x), f(x_0)) = \rho(x, x_0)) \\ &\Rightarrow f(x_0) \in B_\sigma(f(x), \epsilon). \end{aligned}$$

Thus, we have established that $f^{\rightarrow} B_\rho(x, \epsilon) \subset B_\sigma(f(x), \epsilon)$, and so f is continuous. □

Proposition. *If $|| \cdot ||$ is an F-pseudonorm on a vectorspace V, then $d(x, y) = ||x - y||$ defines a metric on V.*

Proof. We verify each of the three properties of metrics.

Claim. *$d(x, y) \geq 0$ for all $x, y \in V$ with equality if and only if $x = y$.*

Proof. For all $x, y \in V$, we have that $d(x, y) = ||x - y|| \geq 0$ by definition of F-pseudonorm. Furthermore,

$$\begin{aligned} d(x, y) = 0 &\Leftrightarrow ||x - y|| = 0 \\ &\Leftrightarrow x - y = 0 \qquad (\text{as } || \cdot || \text{ is an } F\text{-pseudonorm}) \\ &\Leftrightarrow x = y. \end{aligned}$$

□

Claim. *$d(x, y) = d(y, x)$ for all $x, y \in V$.*

Proof. For all $x, y \in V$,

$$\begin{aligned} d(x, y) &= ||x - y|| \\ &= || - 1(y - x)|| \\ &\leq | - 1| \cdot ||y - x|| \qquad (\text{as } || \cdot || \text{ is an } F\text{-pseudonorm}) \\ &= ||y - x|| \\ &= d(y, x). \end{aligned}$$

Similarly,

$$\begin{aligned} d(y, x) &= ||y - x|| \\ &= || - 1(x - y)|| \\ &\leq | - 1| \cdot ||x - y|| \qquad (\text{as } || \cdot || \text{ is an } F\text{-pseudonorm}) \\ &= ||x - y|| \\ &= d(x, y). \end{aligned}$$

Hence, $d(x, y) = d(y, x)$. □

Claim. *$d(x,z) \le d(x,y) + d(y,z)$ for all $x, y, z \in V$.*

Proof. For all $x, y, z \in V$,

$$\begin{aligned} d(x,z) &= ||x - z|| \\ &= ||x + (-1 \cdot z)|| \\ &\le ||x + y|| + ||y + (-1 \cdot z)|| \qquad \text{(as } ||\cdot|| \text{ is an } F\text{-pseudonorm)} \\ &= ||x + y|| + ||y - z|| \\ &\le ||x + (-1 \cdot y)|| + ||(-1 \cdot y) + y|| + ||y - z|| \qquad \text{(as } ||\cdot|| \text{ is an } F\text{-pseudonorm)} \\ &= ||x - y|| + ||y - z|| \\ &= d(x,y) + d(y,z). \end{aligned}$$

□

Therefore, d is a metric on V, as desired. □

Proposition. *(Problem 2C1) Let (M, ρ) be a pseudometric space. The relation $\sim$ defined on M by*

$$x \sim y \text{ if and only if } \rho(x,y) = 0$$

is an equivalence relation.

Proof. We verify each of the three properties of equivalence relations.

Claim. *$\sim$ is reflexive. That is, for all $x \in M$, $x \sim x$.*

Proof. For all $x \in M$, $\rho(x,x) = 0$, as ρ is a pseudometric. Hence, $x \sim x$. □

Claim. *$\sim$ is symmetric. That is, for all $x, y \in M$, $x \sim y$ implies $y \sim x$.*

Proof. For all $x, y \in M$,

$$\begin{aligned} x \sim y \quad &\Rightarrow \quad \rho(x,y) = 0 \\ &\Rightarrow \quad \rho(y,x) = 0 \qquad (\rho \text{ is a pseudometric, and so is symmetric}) \\ &\Rightarrow \quad y \sim x. \end{aligned}$$

□

Claim. *$\sim$ is transitive. That is, for all $x, y, z \in M$, $x \sim y$ and $y \sim z$ together imply $x \sim z$.*

Proof. For all $x, y, z \in M$,

$$\begin{aligned} x \sim y \text{ and } y \sim z \quad &\Rightarrow \quad \rho(x,y) = 0 \text{ and } \rho(y,z) = 0 \\ &\Rightarrow \quad \rho(x,z) = 0 \qquad (\rho \text{ is a pseudometric, so } \rho(x,z) \le \rho(x,y) + \rho(y,z)) \\ &\Rightarrow \quad x \sim z. \end{aligned}$$

□

Therefore, $\sim$ is an equivalence relation on M, as desired. □

Proposition. *(Problem 2C2) If M^* is the set of equivalence classes in M under the equivalence relation $\sim$ and if ρ^* is defined on M^* by*

$$\rho^*([x],[y]) = \rho(x,y),$$

then ρ^* is a well-defined metric on M^*. (The metric space (M^*, ρ^*) is called the metric identification of (M, ρ).)

Proof.

Claim. *ρ^* is a well-defined function on M^*.*

Proof. Let $x_0, x_1 \in [x] \in M^*$ and $y_0, y_1 \in [y] \in M^*$. As $x_0 \sim x_1$ and $y_0 \sim y_1$, we have

$$\rho(x_0, x_1) = \rho(y_0, y_1) = 0.$$

Now,

$$\begin{aligned}
\rho^*([x_0], [y_0]) &= \rho(x_0, y_0) \\
&\leq \rho(x_0, x_1) + \rho(x_1, y_0) \qquad (\rho \text{ is a pseudometric, so we have the triangle inequality}) \\
&\leq \rho(x_0, x_1) + \rho(x_1, y_1) + \rho(y_1, y_0) \qquad (\rho \text{ is a pseudometric, so we have the triangle inequality}) \\
&=\leq \rho(x_0, x_1) + \rho(x_1, y_1) + \rho(y_0, y_1) \qquad (\rho \text{ is a pseudometric, and so is symmetric}) \\
&= \rho(x_1, y_1) \\
&= \rho^*([x_1], [y_1]).
\end{aligned}$$

Similarly,

$$\begin{aligned}
\rho^*([x_1], [y_1]) &= \rho(x_1, y_1) \\
&\leq \rho(x_1, x_0) + \rho(x_0, y_1) \qquad (\rho \text{ is a pseudometric, so we have the triangle inequality}) \\
&\leq \rho(x_1, x_0) + \rho(x_0, y_0) + \rho(y_0, y_1) \qquad (\rho \text{ is a pseudometric, so we have the triangle inequality}) \\
&=\leq \rho(x_0, x_1) + \rho(x_0, y_0) + \rho(y_0, y_1) \qquad (\rho \text{ is a pseudometric, and so is symmetric}) \\
&= \rho(x_0, y_0) \\
&= \rho^*([x_0], [y_0]).
\end{aligned}$$

Hence, $\rho^*([x_0], [y_0]) = \rho^*([x_1], [y_1])$, and so ρ^* is a well-defined function on M^*. □

We proceed by verifying each of the three properties of metrics for ρ^*.

Claim. *$\rho^*([x], [y]) \geq 0$ for all $[x], [y] \in M^*$ with equality if and only if $[x] = [y]$.*

Proof. As ρ is a metric, it is non-negative, and so ρ^* is non-negative. Now, for all $[x], [y] \in M^*$,

$$\begin{aligned}
\rho^*([x], [y]) = 0 \quad &\Leftrightarrow \quad \rho(x, y) = 0 \\
&\Leftrightarrow \quad x \sim y \\
&\Leftrightarrow \quad [x] = [y] \qquad (\text{equivalence classes are either disjoint or they coincide}).
\end{aligned}$$

□

Claim. *$\rho^*([x], [y]) = \rho([y], [x])$ for all $[x], [y] \in M^*$.*

Proof. For all $[x], [y] \in M^*$,

$$\begin{aligned}
\rho^*([x], [y]) &= \rho(x, y) \\
&= \rho(y, x) \qquad (\rho \text{ is a pseudometric, and so is symmetric}) \\
&= \rho^a st([y], [x])
\end{aligned}$$

□

Claim. *$\rho^*([x], [z]) \leq \rho^*([x], [y]) + \rho^*([y], [z])$ for all $[x], [y], [z] \in M^*$.*

Proof. For all $[x], [y], [z] \in M^*$,

$$\begin{aligned}
\rho^*([x], [z]) &= \rho(x, z) \\
&\leq \rho(x, y) + \rho(y, z) \qquad (\rho \text{ is a metric, so we have the triangle inequality}) \\
&= \rho^*([x], [y]) + \rho^*([y], [z])
\end{aligned}$$

□

Therefore, ρ^* is a well-defined metric on M^*, as desired. □

Definition. A normed linear space is a real linear space X such that a number $||x||$, the norm of x, is associated with each $x \in X$, satisfying

(i) $||x|| \geq 0$, and $||x|| = 0$ if and only if $x = 0$;

(ii) $||\alpha x|| = |a| \cdot ||x||$, for all $\alpha \in \mathbb{R}$;

(iii) $||x + y|| \leq ||x|| + ||y||$.

If (i) is replaced by the weaker condition

(i^-) $||x|| \geq 0$ and $||0|| = 0$,

then X is a pseudonormed linear space.

Proposition. *(Problem 2J2) If $|| \cdot ||_1$ and $|| \cdot ||_2$ are pseudonorms on the same linear space X, they give the same open sets (i.e. are equivalent) if and only if there are constants C and C' such that $||x||_1 \leq C \cdot ||x||_2$ and $||x||_2 \leq C' \cdot ||x||_1$, for all $x \in X$.*

Proof. ($\Rightarrow$) For any $x \in X$, define

$$B_1(\vec{0}, r) = \{x \in X \mid ||x||_1 < r\}$$
$$B_2(\vec{0}, r) = \{x \in X \mid ||x||_2 < r\}$$

Now, as $B_1(\vec{0}, r)$ is open with respect to $|| \cdot ||_1$, it is open with respect to $|| \cdot ||_2$ by hypothesis. Hence, there is $\epsilon_2 > 0$ such that $B_2(\vec{0}, \epsilon_2) \subset B_1(\vec{0}, r)$. (Simiarly, we can find $\epsilon_1 > 0$ such that $B_1(\vec{0}, \epsilon_1) \subset B_2(\vec{0}, r)$.) (I fail to see the next step. The chosen ϵ_i give some open ball contained in the larger ball, but there is no guarantee that any x in, say, $B_1(\vec{0}, r)$ can be found in $B_2(\vec{0}, \epsilon_2)$, so I cannot make any claim about $||x||_2$.)

($\Leftarrow$) Let U be open with respect to $|| \cdot ||_1$. That is, for all $x \in U$, there exists $\epsilon > 0$ such that

$$B_1(x, \epsilon) \subset U,$$

where the subscript "1" denotes that distance is computed using $|| \cdot ||$. Now,

$$\begin{aligned}
y \in B_2(x, \frac{\epsilon}{C}) \quad &\Rightarrow \quad ||x - y||_2 < \frac{\epsilon}{C} \\
&\Rightarrow \quad ||x - y||_1 < C \cdot \frac{\epsilon}{C} \qquad \text{(since } ||x - y||_1 \leq C||x - y||_2\text{)} \\
&\Rightarrow \quad ||x - y||_1 < \epsilon \\
&\Rightarrow \quad y \in B_1(x, \epsilon) \\
&\Rightarrow \quad B_2(x, \frac{\epsilon}{C}) \subset B_1(x, \epsilon)
\end{aligned}$$

Hence, U is open with respect to $|| \cdot ||_2$. Similarly, all sets open with respect to $|| \cdot ||_2$ are open with respect to $|| \cdot ||_1$, and so $|| \cdot ||_1$ and $|| \cdot ||_2$ are equivalent. □