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Math 730
Homework 2

Extra Problem

Show that $f : A \to B$ is a bijection if and only if it has a two-sided inverse.

Proof. ($\Rightarrow$) Let f be a bijection. This implies two important facts. Firstly,

$$\begin{aligned} f \text{ bijective} &\Rightarrow f \text{ injective} \\ &\Rightarrow \text{for all } x_0, x_1 \in A,\ x_0 = x_1 \text{ whenever } f(x_0) = f(x_1). \end{aligned}$$

Secondly,

$$\begin{aligned} f \text{ bijective} &\Rightarrow f \text{ surjective} \\ &\Rightarrow \text{for all } y \in A, \text{ there is } x \in A \text{ such that } f(x) = y. \end{aligned}$$

Taken together, we have that

$$f \text{ bijective} \quad \Rightarrow \quad \text{for all } y \in B \text{ there is a unique } x \in A \text{ such that } f(x) = y.$$

In other words, every element of B is of the form $f(x)$ for some unique $x \in A$. Now, define

$$\begin{gathered} g : B \to A \\ g(f(x)) = x \text{ for all } f(x) \in B \end{gathered}$$

We see that, for all $x \in A$,

$$\begin{aligned} (g \circ f)(x) &= g(f(x)) \\ &= x, \end{aligned}$$

so g is a left inverse of f. We also have, for all $f(x) \in B$,

$$\begin{aligned} (f \circ g)(f(x)) &= f(g(f(x)) \\ &= f(x) \text{ (as } g(f(x)) = x \text{ by definition)}, \end{aligned}$$

and so g is a right inverse of f. Therefore, g is a two-sided inverse.

($\Leftarrow$) Let g be a two-sided inverse of f.

First, suppose that f is not an injection. There are $x_1, x_2 \in A$ such that

$$x_1 \neq x_2 \text{ yet } f(x_1) = f(x_2).$$

It follows that

$$\begin{aligned} x_1 &= g(f(x_1)) \\ &= g(f(x_2)) \text{ (as } f(x_1) = f(x_2)) \\ &= x_2 \end{aligned}$$

which is a contradiction with the fact that $x_1 \neq x_2$. Hence, f is an injection.

Next, suppose that f is not a surjection. There exists $y \in B$ such that

$$f(x) \neq y \text{ for all } x \in A.$$

On the other hand,

$$f(g(y)) = y.$$

Hence, there exists an element of A whose image under f is y (namely $g(y)$), which is a contradiction. Hence, f is a surjection.

Finally, as f is both an injection and a surjection, we conclude that f is indeed a bijection, as desired. □

Problem 2B1 (Metrics on $C(\mathbf{I})$)

Let $C(\mathbf{I})$ denote the set of all continuous, real-valued functions on the unit interval $\mathbf{I}$. Show that

$$\rho(f,g) = \sup_{x\in\mathbf{I}} |f(x) - g(x)|$$

is a metric on $C(\mathbf{I})$.

Proof. We verify each of the three properties of metrics.

Claim. *$\rho(f,g) \geq 0$ for all $f, g \in C(\boldsymbol{I})$ with equality if and only if $f = g$.*

Proof. By definition, the ouput of absolute value is always nonnegative, so we have that the supremum of a set of absolute values must also be nonnegative. That is, $\rho(f,g) \geq 0$ for all $f, g \in C(\mathbf{I})$. Now,

$$\begin{aligned} \rho(f,g) = 0 \quad &\Leftrightarrow \quad \sup_{x\in\mathbf{I}} |f(x) - g(x)| \\ &\Leftrightarrow \quad |f(x) - g(x)| = 0 \text{ for all } x \in \mathbf{I} \\ &\Leftrightarrow \quad f(x) = g(x) \text{ for all } x \in \mathbf{I} \\ &\Leftrightarrow \quad f = g \text{ on } \mathbf{I} \end{aligned}$$

□

Claim. *$\rho(f,g) = \rho(g,f)$ for all $f, g \in C(\boldsymbol{I})$.*

Proof. For all $f, g \in C(\mathbf{I})$,

$$\begin{aligned} \rho(f,g) &= \sup_{x\in\mathbf{I}} |f(x) - g(x)| \\ &= \sup_{x\in\mathbf{I}} |g(x) - f(x)| \\ &= \rho(g,f) \end{aligned}$$

□

Claim. *$\rho(f,g) \leq \rho(f,h) + \rho(h,g)$ for al $f, g, h \in C(\boldsymbol{I})$.*

Proof. For all $f, g, h \in C(\mathbf{I})$,

$$\begin{aligned} \rho(f,g) &= \sup_{x\in\mathbf{I}} |f(x) - g(x)| \\ &\leq \sup_{x\in\mathbf{I}} (|f(x) - h(x)| + |h(x) - g(x)|) \quad (\text{as } |\cdot| \text{ is a metric on } \mathbf{I}) \\ &\leq \sup_{x\in\mathbf{I}} |f(x) - h(x)| + \sup_{x\in\mathbf{I}} |h(x) - g(x)| \\ &= \rho(f,h) + \rho(h,g) \end{aligned}$$

□

Therefore, ρ is a metric on $C(\mathbf{I})$, as desired. □

Lemma. *Let f be a non-negative, continuous function on $[0,1]$. If $\int_0^1 f(x)\,dx = 0$, then $f = 0$.*

Proof. We prove the claim above by contrapositive. To that end, choose $x_0 \in [0,1]$ such that $f(x_0) = c > 0$. As f is continuous at x_0, there is $\delta > 0$ such that

$$\begin{aligned} |f(x_0) - f(x)| < c \text{ for all } x \in B(x_0,\delta) \quad &\Rightarrow \quad f(x) > f(x_0) - c \text{ for all } x \in B(x_0,\delta) \\ &\Rightarrow \quad f(x) > 0 \text{ for all } x \in B(x_0,\delta). \end{aligned}$$

Now,

$$\begin{aligned} \int_0^1 f(x)\,dx &\geq \int_{B(x_0,\delta)} f(x)\,dx \\ &\geq \delta \cdot \min\{f(x) \mid x \in B(x_0,\delta)\} \\ &> 0 \end{aligned}$$

thus establishing the contrapositive, as desired. □

Problem 2B2 (Metrics on $C(\mathbf{I})$)

Let $C(\mathbf{I})$ denote the set of all continuous, real-valued functions on the unit interval $\mathbf{I}$. Show that

$$\sigma(f,g) = \int_0^1 |f(x) - g(x)|dx$$

is a metric on $C(\mathbf{I})$.

Proof. We verify each of the three properties of metrics.

Claim. *$\sigma(f,g) \geq 0$ for all $f,g \in C(\boldsymbol{I})$ with equality if and only if $f = g$.*

Proof. By definition, the output of absolute value is always nonnegative, so we have that the integral of the nonnegative function $|f(x) - g(x)|$ is itself nonnegative. That is, $\sigma(f,g) \geq 0$ for all $f,g \in C(\mathbf{I})$. Now, the integral of a nonnegative function is zero if and only if the function is identically zero (call this contention $*$). Hence, we have

$$\begin{aligned} \sigma(f,g) = 0 \quad &\Leftrightarrow \quad \int_0^1 |f(x) - g(x)|dx = 0 \\ &\Leftrightarrow \quad |f(x) - g(x)| = 0 \text{ for all } x \in \mathbf{I} \text{ (by } * \text{)} \\ &\Leftrightarrow \quad f(x) = g(x) \text{ for all } x \in \mathbf{I} \\ &\Leftrightarrow \quad f = g \text{ on } \mathbf{I} \end{aligned}$$

□

Claim. *$\sigma(f,g) = \sigma(g,f)$ for all $f,g \in C(\boldsymbol{I})$.*

Proof. For all $f,g \in C(\mathbf{I})$,

$$\begin{aligned} \sigma(f,g) &= \int_0^1 |f(x) - g(x)|dx \\ &= \int_0^1 |g(x) - f(x)|dx \\ &= \sigma(g,f) \end{aligned}$$

□

Claim. *$\sigma(f,g) \leq \sigma(f,h) + \sigma(h,g)$ for al $f,g,h \in C(\boldsymbol{I})$.*

Proof. For all $f, g, h \in C(\mathbf{I})$,

$$\begin{aligned}
\sigma(f,g) &= \int_0^1 |f(x) - g(x)| dx \\
&\le \int_0^1 (|f(x) - h(x)| + |h(x) - g(x)|) dx \quad \text{(as } |\cdot| \text{ is a metric on } \mathbf{I}\text{)} \\
&= \int_0^1 |f(x) - h(x)| dx + \int_0^1 |h(x) - g(x)| dx \quad \text{(by the linearity of the integral)} \\
&= \sigma(f,h) + \sigma(h,g)
\end{aligned}$$

□

Therefore, σ is a metric on $C(\mathbf{I})$, as desired. □

Problem 2D (Disks are open)

Show that, for any subset A of a metric space (M, d) and any $\epsilon > 0$, the set $B(A, \epsilon)$ is open. (In particular, $B(x, \epsilon)$ is open for each $x \in M$.)

Proof. Let $\epsilon > 0$ be given. Choose $x \in B(A, \epsilon)$. This means that

$$\inf\{d(x, y) \mid y \in A\} < \epsilon$$

This implies that there is some $y_0 \in A$ such that $d(x, y_0) < \epsilon$. Hence, we can find ϵ' such that

$$\epsilon' < \epsilon - d(x, y_0)$$

Claim. $B(x, \epsilon') \subset B(A, \epsilon)$

Proof. Let $x' \in B(x, \epsilon')$. We see that

$$\begin{aligned} d(x', A) &= \inf\{d(x', y) \mid y \in A\} \\ &\leq d(x', y_0) \quad (\text{as } y_0 \in A) \\ &\leq d(x', x) + d(x, y_0) \\ &< \epsilon' + d(x, y_0) \\ &< (\epsilon - d(x, y_0)) + d(x, y_0) \\ &= \epsilon \end{aligned}$$

Hence, $x' \in B(A, \epsilon)$, and so $B(x, \epsilon') \subset B(A, \epsilon)$. □

As $B(x, \epsilon')$ contains x and is contained in $B(A, \epsilon)$, we conclude that $B(A, \epsilon)$ is indeed open. As a special case, letting $A = \{x\}$ shows that, for any $x \in M$, $B(x, \epsilon)$ is open. □

Problem 2E1 (Bounded metrics)

A metric ρ on M is bounded if and only if, for some constant A, $\rho(x, y) \leq A$ for all x and y in M. Show that, if ρ is any metric on M, the distance function

$$\rho^*(x, y) = \min\{\rho(x, y), 1\}$$

is a metric and is also bounded.

Proof. We verify each of the three properties of metrics.

Claim. *$\rho^*(x, y) \geq 0$ for all $x, y \in M$ with equality if and only if $x = y$.*

Proof. As ρ is a metric, $\rho(x, y) \geq 0$ for all $x, y \in M$, and hence $\rho^*(x, y) = \min\{\rho(x, y), 1\} \geq 0$ for all $x, y \in M$. Similarly, $\rho(x, y) = 0$ if and only if $x = y$, and so $\rho^*(x, y) \min\{\rho(x, y), 1\} = 0$ if and only if $x = y$. □

Claim. *$\rho^*(x, y) = \rho^*(x, y)$ for all $x, y \in M$.*

Proof. For all $x, y \in M$,

$$\begin{aligned} \rho^*(x, y) &= \min\{\rho(x, y), 1\} \\ &= \min\{\rho(y, x), 1\} \\ &= \rho^*(y, x) \end{aligned}$$

□

Claim. $\rho^*(x,y) \le \rho^*(x,z) + \rho^*(z,y)$ *for al* $x, y, z \in M$.

Proof. For all $x, y, z \in M$,

$$\begin{aligned}\rho^*(x,y) &= \min\{\rho(x,y), 1\} \\ &\le \min\{\rho(x,z) + \rho(z,y), 1\} \quad \text{(as } \rho \text{ is a metric)} \\ &\le \min\{\rho(x,z), 1\} + \min\{\rho(z,y), 1\} \\ &= \rho^*(x,z) + \rho^*(z,y)\end{aligned}$$

□

Therefore, ρ^* is a metric on M, as desired. Furthermore, we see that ρ^* is bounded by 1. □

Problem 2E2 (Bounded metrics)
A function f is continuous on (M, ρ) if and only if it is continuous on (M, ρ^*).

Proof. One way to define continuity of f is to say that $f^{\leftarrow}(O)$ is open whenever O is open. Hence, it suffices to show that ρ and ρ^* generate the same collection of open sets (if this holds, then we will have that $f^{\leftarrow}(O)$ and O are open with respect to ρ if and only if they are open with respect to ρ^*). To that end, let $A \subset M$.

$$\begin{aligned}A \text{ is open with respect to } \rho \quad &\Rightarrow \quad \text{for all } x \in A, \text{ there is } 0 < \epsilon < 1 \text{ such that } B_\rho(x,\epsilon) \subset A \\ &\Rightarrow \quad \rho(x,y) < \epsilon \text{ for all } y \in B_\rho(x,\epsilon) \\ &\Rightarrow \quad \rho^*(x,y) < \epsilon \text{ for all } y \in B_\rho(x,\epsilon) \quad (\text{as } \epsilon < 1) \\ &\Rightarrow \quad A \text{ is open with respect to } \rho^*\end{aligned}$$

Similarly,

$$\begin{aligned}A \text{ is open with respect to } \rho^* \quad &\Rightarrow \quad \text{for all } x \in A, \text{ there is } 0 < \epsilon \text{ such that } B_{\rho^*}(x,\epsilon) \subset A \\ &\Rightarrow \quad \rho^*(x,y) < \epsilon \text{ for all } y \in B_{\rho^*}(x,\epsilon) \\ &\Rightarrow \quad \rho(x,y) < \epsilon \text{ for all } y \in B_{\rho^*}(x,\epsilon) \\ &\Rightarrow \quad A \text{ is open with respect to } \rho\end{aligned}$$

Therefore, ρ and ρ^* generate the same collection of open sets, and so f is continuous on (M, ρ) if and only if it is continuous on (M, ρ^*). □