Math 730 Homework 15

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1 Definitions

A space X is called Lindelöff if, whenever $\mathcal{G}$ is a collection of open sets such that $\bigcup\{G \mid G \in \mathcal{G}\} = X$ (such a collection is called an open cover), there is a countable subcollection $\{G_n \mid n \in \mathbb{N}\}$ such that $\bigcup\{G_n \mid n \in \mathbb{N}\} = X$.

Call a space X D_2 if distinct points can be put into disjoint clopen (i.e. simultaneously closed and open) sets and D_3 if it is T_0 and has a base of clopen sets.

2 Problem 1

Proposition 2.1. *Every D_3 space is D_2 and Tychonoff.*

Proof. Let x and y be elements of a D_3 space X. Given distinct $x, y \in X$, we can find a basic open (hence, clopen) neighborhood V containing one but not the other. Without loss of generality, let it be that $x \in V$ and $y \notin V$. As V is closed and $y \notin V$, there is a basic open (hence, clopen) neighborhood W that contains y and is disjoint from V. Hence, X is D_2.

To show that X is Tychonoff, we show that it is both T_1 and completely regular. Evidently, D_2 implies T_2, which in turn implies T_1. Now, let A be a closed set in X and let $x \notin A$. We can find disjoint open sets B and V_x such that $A \subset B$ and $x \in V_x$. Define the function $f : B \cup V_x \to \mathbf{I}$ sending B to 0 and V_x to 1. Since B and V_x are disjoint, the inverse image of any subset of $\mathbf{I}$ is either $\emptyset$, B, V_x, X, all of which are open. Hence, f is continuous, and so X is Tychonoff. □

3 Problem 2

Proposition 3.1. *Let $\mathcal{G}$ be an open cover of a D_3, Lindelöff space X. There is a (countable) partition $\mathcal{P}$ of X into clopen sets such that, for each $P \in \mathcal{P}$, there is $G \in \mathcal{G}$ such that $P \subset G$.*

Proof. For all $G \in \mathcal{G}$, G can be represented as a union of basic neighborhoods containing each of its points. Hence, we can construct a clopen cover $\mathcal{G}'$ of X. Since X is Lindelöff, we can assume that $\mathcal{G}'$ is countable. That is, $\mathcal{G}' = \{G_n \mid n \in \mathbb{N}\}$.

Now, for each $n \in \mathbb{N}$, define $P_n = G_n \setminus \bigcup_{i=1}^{n-1} G_i$ and let $\mathcal{P} = \{P_n \mid n \in \mathbb{N}\}$. Evidently, $\mathcal{P}$ is countable with $\bigcup\{P \mid P \in \mathcal{P}\} = X$ and $P_i \cap P_j = \emptyset$ whenever $i \neq j$. Furthermore, for all $n \in \mathbb{N}$, $P_n \subset G_n \subset G$ for some $G \in mathcalG$ (since the G_n were chosen to be basic clopen subsets of the open sets of the cover $\mathcal{G}$). □

Remark 3.2. *Intuitively, the above proposition says we can chop X up into small enough clopen pieces so that each piece fits inside some member of $\mathcal{G}$.*