Math 730 Homework 14

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1 Problem 13D3

For a polynomial P in n real variables, let $Z(P) = \{(x_1, \ldots, x_n) \in \mathbb{R}^n \mid P(x_1, \ldots, x_n) = 0\}$. Let $\mathcal{P}$ be the collection of all such polynomials.

Definition 1.1. *The Zariski topology on $\mathbb{R}^n$ is the one having the set $\{Z(P) \mid P \in \mathcal{P}\}$ as a base for its closed sets.*

Proposition 1.2. *On $\mathbb{R}$, the Zariski topology coincides with the cofinite topology.*

Proof. Let $\mathcal{F}_Z$ denote the closed sets in the Zariski topology and let $\mathcal{F}_{co}$ denote the closed sets in the cofinite topology. We show that $\mathcal{F}_Z = \mathcal{F}_{co}$.

Let $F \in \mathcal{F}_Z$. Observe that $F = \bigcap Z(P_\alpha)$ for some subfamily $\{P_\alpha\} \subset \mathcal{P}$. If the subfamily is merely the zero polynomial, then every real number is a root, and so $F = \mathbb{R}$, which belongs to $\mathcal{F}_{co}$. Otherwise, the number of roots of each polynomial is finite, and so the intersection of these sets of roots is finite. Hence, F is finite, and so $F \in \mathcal{F}_{co}$.

Let $F \in \mathcal{F}_{co}$. If $F = \mathbb{R}$, then F corresponds to the set of roots the zero polynomial. If $F = \emptyset$, then F corresponds to the set of roots of a nonzero, constant polynomial. Otherwise, construct the polynomial $P(x) = \prod_{a \in F}(x - a)$ (since F is finite, this is indeed a polynomial). We see that F corresponds to the set of roots of the polynomial P. In any case, we conclude that $F \in \mathcal{F}_Z$. □

Proposition 1.3. *On $\mathbb{R}^n$ with $n \geq 2$, the Zariski topology does not coincide with the cofinite topology.*

Proof. Consider the polynomial $P(x_1, \ldots, x_n) = x_1$ for any $n \geq 2$. We see that $Z(P) = \{(0, y_2, \ldots, y_n) \mid y_i \in \mathbb{R} \text{ for all } 2 \leq i \leq n\}$, which belongs to $\mathcal{F}_Z$. It does not belong to $\mathcal{F}_{co}$, however, as it is neither finite nor equal to $\mathbb{R}^n$. □

2 Problem 13E2

Definition 2.1. *We say that a is an accumulation point of a set A in a space X provided each neighborhood of a meets A in some point other than a. We say a is a condensation point of A provided each neighborhood of a meets A in uncountably-many points. Let A' denote the set of accumulation points of A and $A^\bullet$ denote the set of condensation points of A.*

Proposition 2.2. *For any subset A of a T_1 space, A' is a closed set.*

Proof. Let $x \notin A'$. There exists a basic neighborhood U of x such that $U \cap A \subset \{x\}$. In particular, this implies that there is an open set $O \subset U$ containing x such that $O \cap A \subset \{x\}$. Now, let $y \in O \setminus \{x\}$. In a T_1 space, singletons are closed, and so $O \setminus \{x\}$ is open. As $O \setminus \{x\}$ is an open neighborhood of y missing A, it follows that $y \notin A'$. As $x \notin A'$ by definition, we have that $O \cap A' = \emptyset$. Hence, A' is closed. □

Remark 2.3. *If the space is merely T_0, the above proposition can fail. For a counterexample, consider $\mathbb{R}$ with the topology whose base is $\{(-\infty, a) \mid a \in \mathbb{R}\}$, which is a T_0 space but not a T_1 space (given $x, y \in \mathbb{R}$, we can find a neighborhood containing the smaller of x and y and excluding the other, but not vice versa). In this case, the derived set of $\{0\}$ is $(0, \infty)$, which is not closed.*

Proposition 2.4. *For any subset A of a topological space, $A^\bullet$ is a closed set with $A^\bullet \subset A'$.*

Proof. Let $x \notin A^\bullet$. There exists a basic neighborhood U of x such that $U \cap A$ is a countable set. Now, for any $y \in U$, there is a neighborhood of y (namely, U itself) that meets A in only countably-many points. Hence, $y \notin A^\bullet$ for all $y \in U$. In other words, U is a neighborhood of x missing $A^\bullet$, and so $A^\bullet$ is closed.

Obviously, if every neighborhood of some point meets A in uncountably-many points, then it meets A at some point other than itself. Hence, $A^\bullet \subset A'$. □

3 Problem 13E4

Proposition 3.1. *Given a set A, let $A^1 = A'$, $A^2 = (A^1)')$, and so on. For any positive integer n, there is a set $A \subset \mathbb{R}$ such that $A, A^1, \ldots, A^{n-1}$ are nonempty and $A^n = \emptyset$.*

Proof. Consider $\mathbb{R}$ with the usual topology. The set $A = \{0\}$ has $A' = \emptyset$. For $n \geq 2$, define

$$A = \left\{ \left(\frac{1}{m_1}, \frac{1}{m_2}, \ldots, \frac{1}{m_{n-1}} \right) \mid m_i \in \mathbb{N} \text{ for all } i \right\}.$$

Observe that

$$\left\{ \left(\frac{1}{m_1}, \frac{1}{m_2}, \ldots, \frac{1}{m_{n-3}}, \frac{1}{m_{n-2}}, 0 \right) \mid m_i \in \mathbb{N} \text{ for all } i \right\} \subset A^1,$$

$$\left\{ \left(\frac{1}{m_1}, \frac{1}{m_2}, \ldots, \frac{1}{m_{n-3}}, 0, 0 \right) \mid m_i \in \mathbb{N} \text{ for all } i \right\} \subset A^2,$$

and so on. By induction, $A^k \neq \emptyset$ for all $k < n$, while $A^n = \emptyset$. □