Math 730 Homework 13

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1 Problem 13D1

For a polynomial P in n real variables, let $Z(P) = \{(x_1, \ldots, x_n) \in \mathbb{R}^n \mid P(x_1, \ldots, x_n) = 0\}$. Let $\mathcal{P}$ be the collection of all such polynomials.

Proposition 1.1. $\{Z(P) \mid P \in \mathcal{P}\}$ *is a base for the closed sets of a topology (the* *Zariski topology**)* *on* $\mathbb{R}^n$.

Proof. We show that $\bigcap_{P \in \mathcal{P}} Z(P) = \emptyset$ and that, for any $P_1, P_2 \in \mathcal{P}$, $Z(P_1) \cup Z(P_2)$ is equal to the intersection of some subfamily of $\{Z(P) \mid P \in \mathcal{P}\}$, thereby establishing that $\{Z(P) \mid P \in \mathcal{P}\}$ is a base for the closed sets of a topology on $\mathbb{R}^n$.

Let $P_0 \in \mathcal{P}$ denote the polynomial in n variables whose output is 1 for any input. As P_0 has no roots, it must be that $Z(P_0) = \emptyset$. Hence, $\bigcap_{P \in \mathcal{P}} Z(P) = \emptyset$.

Let $P_1, P_2 \in \mathcal{P}$. We claim that $Z(P_1) \cup Z(P_2) = Z(P_1 P_2)$, which belongs to $\{Z(P) \mid P \in \mathcal{P}\}$. To see this, observe that

$$\begin{aligned} (x_1, \ldots, x_n) \in Z(P_1) \cup Z(P_2) &\Leftrightarrow P_1(x_1, \ldots, x_n) = 0 \text{ or } P_2(x_1, \ldots, x_n) = 0 \\ &\Leftrightarrow P_1(x_1, \ldots, x_n) \cdot P_2(x_1, \ldots, x_n) = 0 \\ &\Leftrightarrow (x_1, \ldots, x_n) \in Z(P_1 P_2). \end{aligned}$$

Hence, $Z(P_1) \cup Z(P_2)$ is the intersection of a subfamily of $\{Z(P) \mid P \in \mathcal{P}\}$ (namely, $\{Z(P_1 P_2)\}$ itself). $\square$