

Math 730 Homework 12

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1 Problem 13B1

Proposition 1.1. *Any subspace of a T_0 -space is T_0 .*

Proof. Let X be a T_0 -space and let A be a subspace of X . For any distinct $x, y \in A \subset X$, there is a set G that is open in X such that

$$\begin{aligned}x &\in G \text{ and } y \notin G, \text{ or} \\x &\notin G \text{ and } y \in G,\end{aligned}$$

since X is a T_0 -space. Now, consider the set $G \cap A$, which is open in A . It follows that

$$\begin{aligned}x &\in G \text{ and } y \notin G, \text{ or} \\x &\notin G \text{ and } y \in G,\end{aligned}$$

Therefore, A is a T_0 -space. □

Proposition 1.2. *Any subspace of a T_1 -space is T_1 .*

Proof. Let X be a T_1 -space and let A be a subspace of X . For any distinct $x, y \in A \subset X$, there is a set G that is open in X such that

$$x \in G \text{ and } y \notin G,$$

since X is a T_1 -space. Now, consider the set $G \cap A$, which is open in A . It follows that

$$x \in G \cap A \text{ and } y \notin G \cap A.$$

Therefore, A is a T_1 -space. □

2 Problem 13B2

Proposition 2.1. *Any nonempty product space is T_1 if and only if each factor space is T_1 .*

Proof. (\Rightarrow) Let $\prod_{\alpha \in \Gamma} X_\alpha$ be T_1 and let x' and y' be distinct points in some particular X_β . Consider elements $x, y \in \prod_{\alpha \in \Gamma} X_\alpha$ which are equal to x' and y' (respectively) in the β^{th} coordinate and have $x_\alpha = y_\alpha$ for all $\alpha \neq \beta$. As $\prod_{\alpha \in \Gamma} X_\alpha$ is T_1 , it follows that there is an open set $\prod_{\alpha \in \Gamma} G_\alpha$ such that $x \in \prod_{\alpha \in \Gamma} G_\alpha$ and $y \notin \prod_{\alpha \in \Gamma} G_\alpha$. Hence, $x_\alpha \in G_\alpha$ for all $\alpha \in \Gamma$, so, in particular, $x' \in G_\beta$. Now, it must be that $y' \notin G_\beta$, as $y_\alpha = x_\alpha \in G_\alpha$ for all $\alpha \neq \beta$. Hence, the set G_β is an open set in X_β with $x' \in G_\beta$ and $y' \notin G_\beta$. That is, X_β is T_1 . As β was chosen arbitrarily, it follows that each factor space is T_1 .

(\Leftarrow) Let X_α be T_1 for all $\alpha \in \Gamma$ and let x, y be distinct points in $\prod_{\alpha \in \Gamma} X_\alpha$. Since each factor space is T_1 , we can find, for all $\alpha \in \Gamma$, $G_\alpha \subset X_\alpha$ such that $x_\alpha \in G_\alpha$ and $y_\alpha \notin G_\alpha$. It follows that $x \in \prod_{\alpha \in \Gamma} G_\alpha$ and $y \notin \prod_{\alpha \in \Gamma} G_\alpha$. Therefore, $\prod_{\alpha \in \Gamma} X_\alpha$ is T_1 . □