Math 730 Homework 11

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1 Problem 9B1

Definition 1.1. *Let X be a topological space. A* decomposition *$\mathcal{D}$ of X is a collection of disjoint subsets of X whose union is X. If a decomposition $\mathcal{D}$ is endowed with the topology in which $\mathcal{F} \subset \mathcal{D}$ is open if and only if*

$$\bigcup \{F \mid F \in \mathcal{F}\}$$

is open in X, then $\mathcal{D}$ is referred to as a decomposition space of X.

Proposition 1.2. *The process given above for forming the topology on a decomposition space does indeed define a topology.*

Proof. Let τ be the proposed topology on $\mathcal{D}$. We verify that τ satisfies each of the properties of a topology.

Claim 1.3. *If $\mathcal{F}_\alpha$ belongs to τ for all $\alpha \in \Gamma$, then so does $\bigcup_{\alpha \in \Gamma} \mathcal{F}_\alpha$.*

Proof. By definition, each $\mathcal{F}_\alpha$ is itself a union of sets that are open in X. Denote of consituent open sets of $\mathcal{F}_\alpha$ by $F_{\alpha\beta}$ with $\beta \in \Gamma'$. It follows that

$$\begin{aligned} \bigcup_{\alpha \in \Gamma} \mathcal{F}_\alpha &= \bigcup_{\alpha \in \Gamma} \bigcup_{\beta \in \Gamma'} F_{\alpha\beta} \\ &= \bigcup_{\alpha \in \Gamma,\ \beta \in \Gamma'} F_{\alpha\beta}. \end{aligned}$$

Since each of the $F_{\alpha\beta}$ is open in X, their union is also open in X. Therefore, $\bigcup_{\alpha \in \Gamma} \mathcal{F}_\alpha$ belongs to τ. $\square$

Claim 1.4. *If $\mathcal{F}_i$ belongs to τ for all $1 \leq i \leq n$, then so does $\bigcap_{i=1}^n \mathcal{F}_i$.*

Proof. By definition, each $\mathcal{F}_i$ is itself a union of sets that are open in X. Denote of consituent open sets of $\mathcal{F}_i$ by $F_{i\alpha}$ with $\alpha \in \Gamma$. It follows that

$$\bigcap_{i=1}^n \mathcal{F}_i = \bigcap_{i=1}^n \bigcup_{\alpha \in \Gamma} F_{i\alpha}.$$

Since each of the $F_{i\alpha}$ is open in X, their union is also open in X. Since the finite intersetion of open sets is again open, we have that $\bigcap_{i=1}^n \mathcal{F}_i$ belongs to τ, as desired. $\square$

Claim 1.5. *The sets $\emptyset$ and $\mathcal{D}$ belong to τ.*

Proof. We have that

$$\begin{aligned}\bigcup\{F : F \in \emptyset\} &= \bigcup \emptyset \\ &= \emptyset,\end{aligned}$$

which is open in X. Hence, $\emptyset$ belongs to τ.

We also have that

$$\bigcup\{F : F \in \mathcal{D}\} = X,$$

as $\mathcal{D}$ is a decomposition of X. Since X is open in X, we have that $\mathcal{D}$ belongs to τ. □

Therefore, τ is a topology on $\mathcal{D}$. □

2 Problem 9B2

Let $\mathcal{D}$ denote the decomposition space of a topological space X and let P be the natural map from X onto $\mathcal{D}$.

Lemma 2.1. *For any subset $\mathcal{F}$ of $\mathcal{D}$, $P^{\leftarrow}(\mathcal{F}) = \bigcup\{F \mid F \in \mathcal{F}\}$.*

Proof.

$$\begin{aligned}x \in P^{\leftarrow}(\mathcal{F}) &\Leftrightarrow P^{\rightarrow}(x) = F, \text{ some } F \in \mathcal{F} \\ &\Leftrightarrow x \in F, \text{ some } F \in \mathcal{F} \\ &\Leftrightarrow x \in \bigcup\{F \mid F \in \mathcal{F}\}\end{aligned}$$

□

Proposition 2.2. *The topology on $\mathcal{D}$ is the quotient topology induced by P.*

Proof. Let τ denote the topology on the decomposition space $\mathcal{D}$ and let τ_P denote the quotient topology induced by the natural map. It follows that

$$\begin{aligned}\mathcal{O} \in \tau &\Leftrightarrow \mathcal{O} = \{F_\alpha \mid F_\alpha \subset X,\ \alpha \in \Gamma\} \text{ with } \bigcup_{\alpha\in\Gamma} F_\alpha \text{ open in } X \\ &\Leftrightarrow P^{\leftarrow}(\mathcal{O}) \text{ open in } X \qquad\qquad (\text{as } P^{\leftarrow}(\mathcal{O}) = \bigcup_{\alpha\in\Gamma} F_\alpha \text{ by 2.1}) \\ &\Leftrightarrow \mathcal{O} \in \tau_P.\end{aligned}$$

Therefore, $\tau = \tau_P$. □

3 Problem 9C2

Proposition 3.1. *A closed, continuous, onto map need not be open.*

Proof. Consider the function f mapping the interval $[-\pi, 3\pi]$ to the unit circle C by $f(x) = (\cos x, \sin y)$. It is easily seen that f is closed (the image of a closed subinterval of $[-\pi, 3\pi]$ is a closed arc of C), continuous (the preimage of an open arc in C is an open subinterval of $[-\pi, 3\pi]$), and onto ($f^{\rightarrow}[0, 2\pi] = C$).

(I cannot work out how to make use of this counterexample suggested by Willard. It seems that, for any open subinterval O, $f^{\rightarrow}(O)$ is an open arc of C. Hence, a general open set in $[-\pi, 3\pi]$ will simply be the union of open sets in C, which is again open. By the way, I chose $[-\pi, 3\pi]$ as the domain, since any bijection is necessarily both open and closed. Hence, I was hoping to exploit the fact that my choice of domain causes f to fail to be injective to create my counterexample.) □

4 Problem 9H1

Definition 4.1. *Suppose X_α is a topological space and f_α is a map of X_α to a set Y for each $\alpha \in A$. The strong topology coinduced by the maps f_α on Y consists of all sets U in Y such that $f_\alpha^{-1}(U)$ is open in X_α for each $\alpha \in A$.*

Proposition 4.2. *The strong topology is indeed a topology. Moreover, it is the largest topology making each f_α continuous.*

Proof. Let τ be the proposed topology on Y. We verify that τ satisfies each of the properties of a topology.

Claim 4.3. *If O_β belongs to τ for all $\beta \in \Gamma$, then so does $\bigcup_{\beta\in\Gamma} O_\beta$.*

Proof. It follows immediately that

$$\begin{aligned} O_\beta \in \tau \text{ for all } \beta \in \Gamma &\Rightarrow f_\alpha^{-1}(O_\beta) \text{ is open in } X_\alpha \text{ for all } \alpha \in A \text{ and } \beta \in \Gamma \\ &\Rightarrow \bigcup_{\beta\in\Gamma} f_\alpha^{-1}(O_\beta) \text{ is open in } X_\alpha \text{ for all } \alpha \in A \\ &\Rightarrow f_\alpha^{-1}\left(\bigcup_{\beta\in\Gamma} O_\beta\right) \text{ is open in } X_\alpha \text{ for all } \alpha \in A \qquad \left(\text{as } f_\alpha^{-1}\left(\bigcup_{\beta\in\Gamma} O_\beta\right) = \bigcup_{\beta\in\Gamma} f_\alpha^{-1}(O_\beta)\right) \\ &\Rightarrow \bigcup_{\beta\in\Gamma} O_\beta \in \tau. \end{aligned}$$

□

Claim 4.4. *If O_i belongs to τ for all $1 \le i \le n$, then so does $\bigcap_{i=1}^n O_i$.*

Proof. (I am stumped here. The best I can claim is that, for all $\alpha \in A$,

$$f_\alpha^{-1}\left(\bigcap_{i=1}^n O_i\right) \subset \bigcap_{i=1}^n f_\alpha^{-1}(O_i)$$

We have that the righthand side is open in X_α, since each element of the finite intersection is open by hypothesis. I do not see that this forces the lefthand side to be open, as well.) □

Claim 4.5. *The sets $\emptyset$ and Y belong to τ.*

Proof. For all $\alpha \in A$,

$$f_\alpha^{-1}(\emptyset) = \emptyset,$$

which is open in X_α. Hence, $\emptyset \in \tau$.

For all $\alpha \in A$,

$$f_\alpha^{-1}(Y) = X_\alpha,$$

which is open in X_α. Hence, $Y \in \tau$. □

Therefore, τ is indeed a topology on Y. To see that this is the largest topology making each f_α continuous, consider any other topology σ making each f_α continuous. Given any open set O belonging to σ, we have that $f_\alpha^{-1}(O)$ is open for all $\alpha \in A$ (as each f_α is continuous). By definition of openness in τ, we have that $O \in \tau$. Hence, $\sigma \subset \tau$, and so τ is indeed the largest topology making each f_α continuous. □