

Math 730 Homework 10 (Correction 1)

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1 Problem 8D3

Proposition 1.1. *Let X_α be topological spaces containing subsets A_α , respectively, for $\alpha \in \Gamma$. In the product space $\prod_{\alpha \in \Gamma} X_\alpha$, we have*

$$\overline{\prod_{\alpha \in \Gamma} A_\alpha} = \prod_{\alpha \in \Gamma} \overline{A_\alpha}.$$

Proof. Recall that, for any set subset E of a topological space X ,

$$\overline{E} = \{x \in X \mid \text{each basic neighborhood of } x \text{ meets } E\}.$$

It follows that,

$$\begin{aligned} \overline{\prod_{\alpha \in \Gamma} A_\alpha} &= \left\{ x \in \prod_{\alpha \in \Gamma} X_\alpha \mid \text{each basic neighborhood of } x \text{ meets } \prod_{\alpha \in \Gamma} A_\alpha \right\} \\ &= \left\{ x \in \prod_{\alpha \in \Gamma} X_\alpha \mid \text{for all } \alpha \in \Gamma, \text{ each basic neighborhood of } x_\alpha \text{ meets } A_\alpha \right\} \\ &= \left\{ x \in \prod_{\alpha \in \Gamma} X_\alpha \mid \text{for all } \alpha \in \Gamma, \text{ each basic neighborhood of } x_\alpha \in \overline{A_\alpha} \right\} \\ &= \prod_{\alpha \in \Gamma} \overline{A_\alpha}. \end{aligned}$$

□

Remark 1.2. *The proposition above is not true in general for the interior operation. For a counterexample, let $A_\alpha = (0, 1)$ for $\alpha \in \Gamma$ and consider them as subspaces of \mathbb{R}^ω with the usual topology. We have*

$$\begin{aligned} \left(\prod_{\alpha \in \Gamma} (0, 1) \right)^\circ &= \bigcup \left\{ G \mid G \text{ is open, } G \subset \prod_{\alpha \in \Gamma} (0, 1) \right\} \\ &= \emptyset. \end{aligned}$$

To see why this is the case, recall that the open sets in this product topology must equal \mathbb{R} at all but finitely-many coordinates. Hence, the only open set contained in $\prod_{\alpha \in \Gamma} A_\alpha$ is the empty set.

On the other hand, we have

$$\begin{aligned} \prod_{\alpha \in \Gamma} (0, 1)^\circ &= \prod_{\alpha \in \Gamma} (0, 1) \\ &\neq \emptyset. \end{aligned}$$

2 Problem 8H1

Proposition 2.1. *Let X have the weak topology induced by a collection of maps $f_\alpha : X \rightarrow X_\alpha$ for $\alpha \in \Gamma$. If each X_α has the weak topology given by a collection of maps $g_{\alpha\lambda} : X_\alpha \rightarrow Y_{\alpha\lambda}$, for $\lambda \in \Lambda_\alpha$, then X has the weak topology given by the maps $g_{\alpha\lambda} \circ f_\alpha : X \rightarrow Y_{\alpha\lambda}$, for $\alpha \in \Gamma$ and $\lambda \in \Lambda_\alpha$.*

Proof. We first verify that the open sets in X are indeed sufficient to make the functions $g_{\alpha\lambda} \circ f_\alpha$ continuous for all $\alpha \in \Gamma$ and $\lambda \in \Lambda$. To that end, choose an open set $O_{\alpha\lambda} \in Y_{\alpha\lambda}$. It follows that

$$(g_{\alpha\lambda} \circ f_\alpha)^{-1}(O_{\alpha\lambda}) = f_\alpha^{-1}(g_{\alpha\lambda}^{-1}(O_{\alpha\lambda})).$$

Since $g_{\alpha\lambda}$ is continuous, $g_{\alpha\lambda}^{-1}(O_{\alpha\lambda})$ is open. Since f_α is continuous, the preimage of this open set is itself open. Hence, $g_{\alpha\lambda} \circ f_\alpha$ is continuous.

Evidently, the open sets in X are also necessary to make the functions $g_{\alpha\lambda} \circ f_\alpha$ continuous for all $\alpha \in \Gamma$ and $\lambda \in \Lambda$ (i.e. they form the weak topology on X induced by $g_{\alpha\lambda} \circ f_\alpha$). To that end, let $O \in X$. Since the f_α induce the weak topology on X , there exists some O_α such that $f_\alpha^{-1}(O_\alpha) = O$. Furthermore, since the $g_{\alpha\lambda}$ induce the weak topology on X_α , there exists some $O_{\alpha\lambda}$ such that $g_{\alpha\lambda}^{-1}(O_{\alpha\lambda}) = O_\alpha$. Hence,

$$\begin{aligned} (g_{\alpha\lambda} \circ f_\alpha)^{-1}(O_{\alpha\lambda}) &= f_\alpha^{-1}(g_{\alpha\lambda}^{-1}(O_{\alpha\lambda})) \\ &= f_\alpha^{-1}(O_\alpha) \\ &= O. \end{aligned}$$

As the open sets in X are both necessary and sufficient to make the functions $g_{\alpha\lambda} \circ f_\alpha$ continuous for all $\alpha \in \Gamma$ and $\lambda \in \Lambda$, we conclude that X indeed has the weak topology induced by $g_{\alpha\lambda} \circ f_\alpha$. \square