

# Math 730 Homework 10 (Correction 1)

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## 1 Problem 8D3

**Proposition 1.1.** *Let  $X_\alpha$  be topological spaces containing subsets  $A_\alpha$ , respectively, for  $\alpha \in \Gamma$ . In the product space  $\prod_{\alpha \in \Gamma} X_\alpha$ , we have*

$$\overline{\prod_{\alpha \in \Gamma} A_\alpha} = \prod_{\alpha \in \Gamma} \overline{A_\alpha}.$$

*Proof.* Recall that, for any set subset  $E$  of a topological space  $X$ ,

$$\overline{E} = \{x \in X \mid \text{each basic neighborhood of } x \text{ meets } E\}.$$

It follows that,

$$\begin{aligned} \overline{\prod_{\alpha \in \Gamma} A_\alpha} &= \left\{ x \in \prod_{\alpha \in \Gamma} X_\alpha \mid \text{each basic neighborhood of } x \text{ meets } \prod_{\alpha \in \Gamma} A_\alpha \right\} \\ &= \left\{ x \in \prod_{\alpha \in \Gamma} X_\alpha \mid \text{for all } \alpha \in \Gamma, \text{ each basic neighborhood of } x_\alpha \text{ meets } A_\alpha \right\} \\ &= \left\{ x \in \prod_{\alpha \in \Gamma} X_\alpha \mid \text{for all } \alpha \in \Gamma, \text{ each basic neighborhood of } x_\alpha \in \overline{A_\alpha} \right\} \\ &= \prod_{\alpha \in \Gamma} \overline{A_\alpha}. \end{aligned}$$

□

**Remark 1.2.** *The proposition above is not true in general for the interior operation. For a counterexample, let  $A_\alpha = (0, 1)$  for  $\alpha \in \Gamma$  and consider them as subspaces of  $\mathbb{R}^\omega$  with the usual topology. We have*

$$\begin{aligned} \left( \prod_{\alpha \in \Gamma} (0, 1) \right)^\circ &= \bigcup \left\{ G \mid G \text{ is open, } G \subset \prod_{\alpha \in \Gamma} (0, 1) \right\} \\ &= \emptyset. \end{aligned}$$

To see why this is the case, recall that the open sets in this product topology must equal  $\mathbb{R}$  at all but finitely-many coordinates. Hence, the only open set contained in  $\prod_{\alpha \in \Gamma} A_\alpha$  is the empty set.

On the other hand, we have

$$\begin{aligned} \prod_{\alpha \in \Gamma} (0, 1)^\circ &= \prod_{\alpha \in \Gamma} (0, 1) \\ &\neq \emptyset. \end{aligned}$$

## 2 Problem 8H1

**Proposition 2.1.** *Let  $X$  have the weak topology induced by a collection of maps  $f_\alpha : X \rightarrow X_\alpha$  for  $\alpha \in \Gamma$ . If each  $X_\alpha$  has the weak topology given by a collection of maps  $g_{\alpha\lambda} : X_\alpha \rightarrow Y_{\alpha\lambda}$ , for  $\lambda \in \Lambda_\alpha$ , then  $X$  has the weak topology given by the maps  $g_{\alpha\lambda} \circ f_\alpha : X \rightarrow Y_{\alpha\lambda}$ , for  $\alpha \in \Gamma$  and  $\lambda \in \Lambda_\alpha$ .*

*Proof.* We first verify that the open sets in  $X$  are indeed sufficient to make the functions  $g_{\alpha\lambda} \circ f_\alpha$  continuous for all  $\alpha \in \Gamma$  and  $\lambda \in \Lambda$ . To that end, choose an open set  $O_{\alpha\lambda} \in Y_{\alpha\lambda}$ . It follows that

$$(g_{\alpha\lambda} \circ f_\alpha)^{-1}(O_{\alpha\lambda}) = f_\alpha^{-1}(g_{\alpha\lambda}^{-1}(O_{\alpha\lambda})).$$

Since  $g_{\alpha\lambda}$  is continuous,  $g_{\alpha\lambda}^{-1}(O_{\alpha\lambda})$  is open. Since  $f_\alpha$  is continuous, the preimage of this open set is itself open. Hence,  $g_{\alpha\lambda} \circ f_\alpha$  is continuous.

Evidently, the open sets in  $X$  are also necessary to make the functions  $g_{\alpha\lambda} \circ f_\alpha$  continuous for all  $\alpha \in \Gamma$  and  $\lambda \in \Lambda$  (i.e. they form the weak topology on  $X$  induced by  $g_{\alpha\lambda} \circ f_\alpha$ ). To that end, let  $O \in X$ . Since the  $f_\alpha$  induce the weak topology on  $X$ , there exists some  $O_\alpha$  such that  $f_\alpha^{-1}(O_\alpha) = O$ . Furthermore, since the  $g_{\alpha\lambda}$  induce the weak topology on  $X_\alpha$ , there exists some  $O_{\alpha\lambda}$  such that  $g_{\alpha\lambda}^{-1}(O_{\alpha\lambda}) = O_\alpha$ . Hence,

$$\begin{aligned} (g_{\alpha\lambda} \circ f_\alpha)^{-1}(O_{\alpha\lambda}) &= f_\alpha^{-1}(g_{\alpha\lambda}^{-1}(O_{\alpha\lambda})) \\ &= f_\alpha^{-1}(O_\alpha) \\ &= O. \end{aligned}$$

As the open sets in  $X$  are both necessary and sufficient to make the functions  $g_{\alpha\lambda} \circ f_\alpha$  continuous for all  $\alpha \in \Gamma$  and  $\lambda \in \Lambda$ , we conclude that  $X$  indeed has the weak topology induced by  $g_{\alpha\lambda} \circ f_\alpha$ .  $\square$