Austin Mohr
Math 704
Homework 6

Problem 1

Integrability of f on $\mathbb{R}$ does not necessarily imply the convergence of $f(x)$ to 0 as $x \to \infty$.

a. There exists a positive continuous function f on $\mathbb{R}$ so that f is integrable on $\mathbb{R}$, but yet $\limsup_{x\to\infty} f(x) = \infty$.

Proof. Define the function f to take on the value n if $x \in [n, n+\frac{1}{n^3})$ for $n \geq 2$. Elsewhere, the function is zero except for the line segments required to make the function continuous. We define these segments in such a that the graph resembles a sequence of trapezoids with height n and bases of length $\frac{1}{n^3}$ and $\frac{3}{n^3}$.

By construction, f is positive and continuous on $\mathbb{R}$. Now,

$$\begin{aligned}\int_{-\infty}^{\infty} f dx &= \sum_{n=2}^{\infty} n(\frac{\frac{1}{n^3}+\frac{3}{n^3}}{2}) \\ &= \sum_{n=2}^{\infty} \frac{2}{n^2} \\ &< \infty\end{aligned}$$

Hence, f is integrable. The fact that $\limsup_{x\to\infty} f(x) = \infty$ is also immediate (for any N, there are infinitely-many $x \in (N, \infty)$ such that $f(x) > N$), so the claim is proven. □

b. However, if we assume that f is uniformly continuous on $\mathbb{R}$ and integrable, then $\limsup_{|x|\to\infty} f(x) = 0$.

Proof. Suppose, for the sake of contradiction, that $\limsup_{x\to\infty} f(x) = c > 0$ (we consider first only the case where $x \to +\infty$). Choose some d so that $0 < d < c$. Then, there is a sequence $x_1, x_2, \ldots$ with each x_i far apart (to be made precise later) so that $f(x_i) \geq d$ for all i. Choose ϵ_0 so that $0 < \epsilon_0 < d$. Since f is uniformly continuous, there is some $\delta_0 > 0$ so that, for each x_i, $|f(x_i) - f(y)| < \epsilon_0$ for all $y \in N(x_i, \delta_0)$. Since $f(x_i) \geq d > \epsilon_0$, we have that $f(y) > \epsilon_0$ for all $y \in N(x_i, \delta_0)$. Hence, the area contributed by the function over the interval $N(x_i, \delta_0)$ is at least $2\delta_0\epsilon_0$. Now, if we choose the x_i far enough apart so that each of the $N(x_i, \delta_0)$ are disjoint, we have that

$$\int_0^{\infty} f(x) \geq \sum_{n=1}^{\infty} 2\delta_0\epsilon_0 = \infty$$

which contradicts the fact that f is integrable.

We can force the same contradiction when $x \to -\infty$, and so we conclude that $\limsup_{|x|\to\infty} f(x)dx = 0$. □

Problem 2

Suppose $f \geq 0$, and let $E_{2^k} = \{x \mid f(x) > 2^k\}$ and $F_k = \{x \mid 2^k < f(x) \leq 2^{k+1}\}$. If f is finite almost everywhere, then

$$\bigcup_{k=-\infty}^{\infty} F_k = \{f(x) > 0\},$$

and the sets F_k are disjoint.

Proof. Since f is a function, it has a unique output for each input x. Hence, $2^k < f(x) \leq 2^{k+1}$ for a *single* value of k. That is, the F_k are disjoint.

($\subseteq$) Let $x \in \bigcup_{k=-\infty}^{\infty} F_k$. Then $2^k < f(x) \leq 2^{k+1}$ for some k, so $f(x) \neq 0$. Since $f \geq 0$, we have that $f(x) > 0$. That is, $x \in \{f(x) > 0\}$.

($\supseteq$) Let $x \in \{f(x) > 0\}$. Then $f(x) > 0$, and so $2^k < f(x) \leq 2^{k+1}$ for some k. That is, $x \in \bigcup_{k=-\infty}^{\infty} F_k$. □

Prove that f is integrable if and only if

$$\sum_{k=-\infty}^{\infty} 2^k m(F_k) < \infty$$

if and only if

$$\sum_{k=-\infty}^{\infty} 2^k m(E_{2^k}) < \infty$$

Proof. (i $\Rightarrow$ ii) Suppose f is integrable. Let $x \in F_k$ for some k. By definition of F_k, $2^k < f(x)$. Since the F_k are disjoint, it follows that

$$\begin{aligned}
\infty &> \int f dx \\
&> \sum_{k=-\infty}^{\infty} \int 2^k \chi_{F_k} dx \\
&= \sum_{k=-\infty}^{\infty} 2^k \int \chi_{F_k} dx \\
&= \sum_{k=-\infty}^{\infty} 2^k m(F_k)
\end{aligned}$$

(ii $\Rightarrow$ i) Suppose $\sum_{k=-\infty}^{\infty} 2^k m(F_k) < \infty$. Then $2 \sum_{k=-\infty}^{\infty} 2^k m(F_k) < \infty$. Let $x \in F_k$ for some k. By definition of F_k, $2^{k+1} \geq f(x)$. Since the F_k are disjoint, it follows that

$$\begin{aligned}
\int f dx &< \sum_{k=-\infty}^{\infty} \int 2^{k+1} \chi_{F_k} dx \\
&= \sum_{k=-\infty}^{\infty} 2^{k+1} \int \chi_{F_k} dx \\
&= \sum_{k=-\infty}^{\infty} 2^{k+1} m(F_k) \\
&< \infty
\end{aligned}$$

(ii ⇔ iii) Suppose $\sum_{k=-\infty}^{\infty} 2^k m(F_k) < \infty$. Observe that $E_{2^k} = \bigcup_{n \geq k} F_n$. Since the F_n are disjoint and measurable, it follows that

$$\begin{aligned} m(E_{2^k}) &= m(\bigcup_{n\geq k} F_n) \\ &= \sum_{n\geq k} m(F_n) \end{aligned}$$

So

$$\begin{aligned} \sum_{k=-\infty}^{\infty} 2^k m(E_{2^k}) &= \sum_{k=-\infty}^{\infty} \sum_{n\geq k} 2^k m(F_n) \\ &= \sum_{n=-\infty}^{\infty} \sum_{k=-\infty}^{n} 2^k m(F_n) \\ &= \sum_{n=-\infty}^{\infty} m(F_n) \sum_{k=-\infty}^{n} 2^k \\ &= \sum_{n=-\infty}^{\infty} 2^{n+1} m(F_n) \\ &= 2 \sum_{n=-\infty}^{\infty} 2^n m(F_n) \end{aligned}$$

Hence, if either of $\sum_{k=-\infty}^{\infty} 2^k m(E_{2^k})$ or $\sum_{k=-\infty}^{\infty} 2^k m(F_k)$ is finite, then the other is also finite. □

Use this result to verify the following assertions. Let

$$f(x) = \begin{cases} |x|^{-a} & \text{if } |x| \leq 1 \\ 0 & \text{otherwise} \end{cases}$$

and

$$g(x) = \begin{cases} |x|^{-b} & \text{if } |x| > 1 \\ 0 & \text{otherwise} \end{cases}$$

Then f is integrable on $\mathbb{R}^d$ if and only if $a < d$; also g is integrable on $\mathbb{R}^d$ if and only if $b > d$.

Proof. Let $x \in F_k$. Then

$$\begin{aligned} 2^k &< |x|^{-a} &\leq 2^{k+1} \\ 2^{\frac{-k}{a}} &> |x| &\geq 2^{\frac{-k-1}{a}} \end{aligned}$$

Hence

$$\begin{aligned} m(B(0; 2^{\frac{-k}{a}})) &> m(F_k) &\geq m(B(0; 2^{\frac{-k-1}{a}})) \\ v_d 2^{\frac{-dk}{a}} &> m(F_k) &\geq v_d 2^{\frac{-d(k+1)}{a}} \end{aligned}$$

where v_d is the volume of the unit ball. Now

$$\sum_{k=-\infty}^{\infty} 2^k m(F_k) = \sum_{k=-\infty}^{0} 2^k m(F_k) + \sum_{k=1}^{\infty} 2^k m(F_k)$$

Since $|x| \le 1$, $m(F_k) \le v_d$ for all k. Hence

$$\begin{aligned} \sum_{k=-\infty}^{\infty} 2^k m(F_k) &\le v_d \sum_{k=-\infty}^{0} 2^k + \sum_{k=1}^{\infty} 2^k m(F_k) \\ &= 2v_d + \sum_{k=1}^{\infty} 2^k m(F_k) \end{aligned}$$

So it suffices to show that $\sum_{k=1}^{\infty} 2^k m(F_k)$ converges.

$$\begin{aligned} \sum_{k=1}^{\infty} 2^k (v_d 2^{\frac{-d(k+1)}{a}}) \quad &\le \sum_{k=1}^{\infty} 2^k m(F_k) \quad < \sum_{k=1}^{\infty} 2^k (v_d 2^{\frac{-dk}{a}}) \\ v_d 2^{\frac{-d}{a}} \sum_{k=1}^{\infty} 2^{k(1-\frac{d}{a})} \quad &\le \sum_{k=1}^{\infty} 2^k m(F_k) \quad < v_d \sum_{k=1}^{\infty} 2^{k(1-\frac{d}{a})} \end{aligned}$$

We see that the upper and lower bounds converge if and only if $0 < a < d$, forcing the convergence of $\sum_{k=-\infty}^{\infty} 2^k m(F_k)$, which in turn implies that f is integrable.

Let $x \in E_{2^k}$. Then

$$\begin{aligned} 2^k &< |x|^{-b} \\ 2^{\frac{-k}{b}} &> |x| \end{aligned}$$

Observe also that E_{2^k} is empty for $k \le 0$. Hence, $E_{2^k} = B(0; 2^{\frac{-k}{b}}) \setminus B(0;1)$, and so $m(E_{2^k}) = v_d 2^{\frac{-dk}{b}} - v_d$. Now

$$\begin{aligned} \sum_{k=-\infty}^{\infty} 2^k m(E_{2^k}) &= \sum_{k=-\infty}^{-1} 2^k m(E_{2^k}) \\ &= \sum_{k=-\infty}^{-1} 2^k (v_d 2^{\frac{-dk}{b}} - v_d) \\ &= v_d \sum_{k=-\infty}^{-1} 2^{k(1-\frac{d}{b})} - v_d \sum_{k=-\infty}^{-1} 2^k \end{aligned}$$

which converges if and only if $b > d$, which in turn implies that f is integrable.

□

Problem 3

a. Prove that if f is integrable on $\mathbb{R}^d$, real-valued, and $\int_E f(x)dx \ge 0$ for every measurable E, then $f(x) \ge 0$ a.e. x.

Proof. Define F to be the set $\{x \mid f(x) < 0\}$. Since f is integrable, f is measurable, and so F is measurable. We claim that $m(F) = 0$.

Since F is measurable, we have that $\int_F f(x)dx \geq 0$ by hypothesis. Now, observe that, for any $n \geq 1$, $nf\chi_F \leq f$. It follows that

$$\begin{aligned}\int nf\chi_F dx &\leq \int f dx \\ n\int_F f dx &\leq \int f dx \\ \int_F f dx &\leq \frac{1}{n}\int f dx \\ \int_F f dx &\leq 0\end{aligned}$$

Hence, $\int_F f dx = 0$. Since $f(x) < 0$ for all $x \in F$, we conclude that $m(F) = 0$. That is, $f(x) \geq 0$ almost everywhere.

□

b. As a result, if $\int_E f(x)dx = 0$ for every measurable E, then $f(x) = 0$ a.e.

Proof. From the first part, we see that $f(x) \geq 0$ almost everywhere. Let G be the set $\{x \mid f(x) > 0\}$. It suffices to show that $m(G) = 0$.

As before, f is measurable, so G is measurable. By hypothesis, we have that $\int_G f dx = 0$. Since $f(x) > 0$ for all $x \in G$, we conclude that $m(G) = 0$. Hence, the set of x so that $f(x) \neq 0$ has measure 0. That is, $f(x) = 0$ almost everywhere. □

Problem 4

a. Let $a_n, b_n \in \mathbb{R}$ such that $a_n \to a \in \mathbb{R}$. Prove that

$$\liminf(a_n + b_n) = a + \liminf b_n$$

Proof. Since $a_n \to a$, we have $a = \lim_{n\to\infty} a_n = \limsup_{n\to\infty} a_n = \liminf_{n\to\infty} a_n$. Since $\limsup(a_n) = -\liminf(-a_n)$, we have

$$\begin{aligned}\liminf(a_n + b_n) &= -\limsup(-a_n - b_n) \\ &\geq -\limsup(-a_n) - \limsup(-b_n) \\ &= \liminf a_n + \liminf b_n \\ &= a + \liminf b_n\end{aligned}$$

Now, construct a subsequence (b_{n_k}) of (b_n) with $\lim_{k\to\infty} b_{n_k} = \liminf_{n\to\infty} b_n$. Let (a_{n_k}) be the subsequence induced by the indices chosen for (b_{n_k}).

$$\begin{aligned}a + \liminf b_n &= \liminf a_n + \liminf b_n \\ &= \lim a_{n_k} + \lim b_{n_k} \\ &= \lim(a_{n_k} + b_{n_k}) \\ &\geq \liminf(a_n + b_n)\end{aligned}$$

Therefore, $\liminf(a_n + b_n) = a + \liminf b_n$. □

b. Let f, f_n be integrable functions. Assume $f_n(x) arrow f(x)$ a.e. and $\int |f_n|dx \to \int |f|dx$. Prove that $\int |f_n - f|dx \to 0$.

Proof. Define the function g_n to be $|f| + |f_n| - |f - f_n|$. Then $g_n \to 2|f|$ as $n \to \infty$. By Fatou's lemma

$$\int g dx \leq \liminf \int g_n dx$$

Hence

$$\begin{aligned}
2\int |f|\,dx &\le \liminf \int (|f| + |f_n| - |f - f_n|)\,dx \\
&= \liminf(\int |f|\,dx + \int |f_n|\,dx - \int |f - f_n|\,dx) \\
&= \int |f|\,dx + \liminf(\int |f_n|\,dx - \int |f - f_n|\,dx) \\
&= \int |f|\,dx + \int |f|\,dx + \liminf(-\int |f - f_n|\,dx) \qquad \text{(by part (a))}
\end{aligned}$$

Now,

$$\begin{aligned}
0 &\le \liminf(-\int |f - f_n|\,dx) \\
0 &\ge \limsup \int |f - f_n|\,dx \\
&\ge \lim_{n\to\infty} \int |f - f_n| dx
\end{aligned}$$

Therefore, as $n \to \infty$, $\int |f - f_n|\,dx \to 0$. □