Austin Mohr
Math 704
Homework 5

Problem 1

Consider the exterior Lebesgue measure m^* introduced in Chapter 1. Prove that a set E in $\mathbb{R}^d$ is Carathéodory measurable if and only if E is Lebesgue measurable in the sense of Chapter 1.

Proof. ($\Rightarrow$) Suppose E is Carathéodory measurable. Consider first the case where E has finite measure. We can find open sets O_n so that $m^*(O_n) \leq m(E) + \frac{1}{n}$. Define G to be the G_δ set $\bigcap_{n=1}^{\infty} O_n$. Since $E \subseteq G \subseteq O_n$ for all n, we have that $m^*(E) \leq m^*(G) \leq m^*(O_n) \leq m^(E) + \frac{1}{n}$ for all n. Hence, $m^*(G) = m^*(E)$. Now, since E is Carathéodory measurable, we have

$$\begin{aligned} m^*(G) &= m^*(E \cap G) + m^*(E^c \cap G) \\ m^*(E) &= m^*(E) + m^*(G - E) \qquad \text{(since } m^*(E) = m^*(G) \text{ and } E \subseteq G\text{)} \\ 0 &= m^*(G - E) \end{aligned}$$

We see that E differs from a G_δ set by a set of measure 0. Hence, E is Lebesgue measurable.

Now consider the case where E has infinite measure. Define E_n to be the set $E \cap [-n, n]^d$. For all n, we see that E_n has finite Carathéodory measure, and so is Lebesgue measurable by the previous case. Now, E is the countable union of the Lebesgue measurable E_n, so E is Lebesgue measurable.

($\Leftarrow$) Suppose E is Lebesgue measurable and let $A \subseteq \mathbb{R}^d$ be given. Using the same method as before, we can construct a G_δ set G with $A \subseteq G$ and $m^*(A) = m^*(G)$. Now, observe that G is the union of disjoint sets $E \cap G$ and $E^c \cap G$. Since both E and G are Lebesgue measurable, we have

$$m^*(A) = m^*(G) = m^*(E \cap G) + m^*(E^c \cap G) \tag{1}$$

Now, $m^*(A) \leq m^*(E \cap A) + m^*(E^c \cap A)$ by subadditivity. Since $A \subseteq G$, we see that $m^*(E \cap G) \geq m^*(E \cap A)$ and $m^*(E^c \cap G) \geq m^*(E^c \cap A)$. Hence, $m^*(A) = m^*(G) = m^*(E \cap G) + m^*(E^c \cap G) \geq m^*(E \cap A) + m^*(E^c \cap A)$. Therefore, $m^*(A) = m^*(E \cap A) + m^*(E^c \cap A)$, and so E is Carathéodory measurable. □

Problem 2 (Tchebychev Inequality)

Suppose $f \geq 0$ and f is integrable. If $\alpha > 0$ and $E_\alpha = \{x \mid f(x) > \alpha\}$, prove that

$$m(E_\alpha) \leq \tfrac{1}{\alpha} \int f.$$

Proof. Since f is integrable, it is measurable. This implies, in particular, that E_α is measurable, and so we have that $m(E_\alpha) = \int \chi_{E_\alpha}$. Now, from the definition of E_α, we have that $0 \leq \alpha \chi_{E_\alpha} \leq f$, which in turn gives $\alpha \int \chi_{E_\alpha} \leq \int f$. By our previous observation, we can replace $\int \chi_{E_\alpha} dx$ with $m(E_\alpha)$ to get $m(E_\alpha) \leq \frac{1}{\alpha} \int f$, as desired. □