Austin Mohr
Math 704
Homework 3

Problem 1

Show that there exist closed sets A and B with $m(A) = m(B) = 0$, but $m(A + B) > 0$:

a. In $\mathbb{R}$, let $A = \mathcal{C}$ (the Cantor set), $B = \frac{\mathcal{C}}{2}$.

Proof. Recall that the Cantor set (here, A) contains no open interval, and so has measure 0. Similarly, B has measure 0. Recall also that an element belongs to A if and only if it has a ternary expansion using only 0s and 2s. Hence, an element belongs to B if and only if it has a ternary expansion using only 0s and 1s. Now, let $x \in [0, 1]$. We choose $a \in A$ and $b \in B$ such that $x = a + b$ (and so show that $x \in A + B$) as follows:

If the kth digit of x is 0, specify that the kth digit of a is 0 and the kth digit of b is 0.
If the kth digit of x is 1, specify that the kth digit of a is 0 and the kth digit of b is 1.
If the kth digit of x is 2, specify that the kth digit of a is 2 and the kth digit of b is 0.

Hence, $A + B \supset [0, 1]$, and so by problem 4a, we have that $A + B$ is measurable (since $A + B$ contains a subset of nonzero measure). Furthermore, $m(A + B) \geq m([0, 1]) = 1$. □

b. In $\mathbb{R}^2$, observe that if $A = I \times \{0\}$ and $B = \{0\} \times I$ (where $I = [0, 1]$), then $A + B = I \times I$.

Proof. Lines in $\mathbb{R}^2$ have measure 0, since, in the limit, a covering a closed cubes will have sides of length 0. Hence, A and B both have measure 0. Now, consider

$$I \times I = \{(a, b) \mid a, b \in [0, 1]\}$$

This is a closed cube with sides of length 1, and is therefore measurable with measure 1. □

Problem 2

Prove that there is a continuous function that maps a Lebesgue measurable set to a non-measurable set.

Proof. Consider the function $F : \mathcal{C} \to [0, 1]$ defined previously. We have already shown that F is continuous and surjective. Now, let N be the non-measurable set described in the text, and let $\mathcal{C}'$ denote the preimage of N under F. Since F is surjective, we know that $\mathcal{C}'$ is nonempty. Furthermore, $\mathcal{C}' \subseteq \mathcal{C}$ since $N \subseteq [0, 1]$. Now, consider the function

$$G : \mathcal{C}' \to N$$
$$G(x) = F(x) \text{ for all } x \in \mathcal{C}'$$

(i.e. the function F restricted to the domain $\mathcal{C}'$). G is surjective by definition of $\mathcal{C}'$. Furthermore, since F is continuous, G is continuous. Finally, we see that $m(\mathcal{C}') \leq m(\mathcal{C}) = 0$, since $\mathcal{C}' \subseteq \mathcal{C}$. Therefore, G is a continuous function mapping a Lebesgue measurable set onto a non-measurable set. □

Problem 3

Let E be a subset of $\mathbb{R}$ with $m^*(E) > 0$. Prove that for each $0 < \alpha < 1$, there exists an open interval I so that

$$m^*(E \cap I) \geq \alpha m^*(I)$$

Proof. Choose $\mathcal{O} \supset E$ such that $m^*(E) \geq \alpha m^*(\mathcal{O})$ (this can always be done). We can write

$$\mathcal{O} = \bigcup_{i=1}^{\infty} \mathcal{O}_i,\ \mathcal{O}_i \text{ open and disjoint}$$

and hence

$$\begin{aligned}E &= E \cap \mathcal{O}\\ &= E \cap \bigcup_{i=1}^{\infty} \mathcal{O}_i\\ &= \bigcup_{i=1}^{\infty}(E \cap \mathcal{O}_i)\end{aligned}$$

Now, suppose that $m^*(E \cap \mathcal{O}_i) < \alpha m^*(\mathcal{O}_i)$ for all i. Then

$$\begin{aligned}m^*(E) &= m^*(\bigcup_{i=1}^{\infty}(E \cap \mathcal{O}_i))\\ &= \sum_{i=1}^{\infty} m^*(E \cap \mathcal{O}_i) && \text{(by the disjointness of the } E \cap \mathcal{O}_i\text{)}\\ &< \alpha \sum_{i=1}^{\infty} m^*(\mathcal{O}_i) && \text{(by hypothesis)}\\ &= \alpha m^*(\mathcal{O}) && \text{(by disjointness of the } \mathcal{O}_i\text{)}\end{aligned}$$

which is a contradiction with the fact that $m^*(E) \geq \alpha m^*(\mathcal{O})$. Hence, it must be that, for some k, $m^*(E \cap \mathcal{O}_k) \geq \alpha m^*(\mathcal{O}_k)$, which proves the claim. ☐

Problem 4

Let $\mathcal{N}$ denote the non-measurable subset of $I = [0, 1]$ constructed at the end of Section 3.

a. Prove that if E is a measurable subset of $\mathcal{N}$, then $m(E) = 0$.

Proof. Let $\{r_k\}_{k=1}^{\infty}$ be an enumeration of the rationals in the interval $[-1, 1]$ and let $E_k = E + r_k$ for each k. Since $E \subseteq N$, $E_k \subseteq N_k$ for each k. Since each of the N_k are pairwise disjoint, each of the E_k are pairwise disjoint. Now, the Lebesgue measure is translation invariant, so $m(E_k) = m(E)$ for each k. We also have that $\bigcup_{k=1}^{\infty} E_k \subseteq \bigcup_{k=1}^{\infty} N_k \subseteq [-1, 2]$. It follows that

$$\begin{aligned}\sum_{k=1}^{\infty} m(E_k) &= m(\bigcup_{k=1}^{\infty} E_k) && \text{(by the disjointness of the } E_k\text{)}\\ &\leq 3 && \text{(since } \bigcup_{k=1}^{\infty} E_k \subseteq [-1, 2]\text{)}\end{aligned}$$

But $m(E_k) = m(E)$ for each k. Hence

$$\begin{aligned}3 &\geq \sum_{k=1}^{\infty} m(E_k)\\ &= \sum_{k=1}^{\infty} m(E)\end{aligned}$$

which implies that $m(E) = 0$. ☐

b. If G is a subset of $\mathbb{R}$ with $m^*(G) > 0$, prove that a subset of G is non-measurable.

Proof. Since $m(G) > 0$, we can find for any $\epsilon > 0$ a closed interval $[a, b] \subseteq G$ with $m(G \setminus [a, b]) \leq \epsilon$. Now, consider the set $G - a$ (G translated by $-a$ units). Since the Lebesgue measure is translation invariant, $m(G - a) = m(G)$. Furthermore, the interval $[0, b - a] \subseteq G - a$. Let $A = [0, b - a] \cap N$. Observe that $A \subseteq G$. Suppose A is measurable. Since $A \subseteq N$, $m(A) = 0$ by part (a). It follows that

$$\begin{aligned} m(G) &= m(G - a) \\ &= m((G - a) \setminus A) + m(A) \\ &\leq \epsilon + 0 \\ &= \epsilon \end{aligned}$$

Since ϵ can be chosen arbitrarily small, we conclude that $m(G) = 0$, which is a contradiction. Hence, it must be that A is non-measurable. □