Austin Mohr
Math 704
Homework 2

Problem 1 (Cantor-like Sets)

Construct a closed set $\hat{\mathcal{C}}$ so that at the kth stage of the construction one removes 2^{k-1} centrally situated open intervals each of length l_k with

$$l_1 + 2l_2 + \cdots + 2^{k-1}l_k < 1$$

a) If l_j are chosen small enough, then

$$\sum_{k=1}^{\infty} 2^{k-1}l_k < 1$$

In this case, show that $m(\hat{\mathcal{C}}) > 0$, and in fact,

$$m(\hat{\mathcal{C}}) = 1 - \sum_{k=1}^{\infty} 2^{k-1}l_k$$

Proof. First, we claim that $\hat{\mathcal{C}}$ is measurable. Denote by O_k the union of the open sets removed from $[0,1]$ at step k of the construction. Since the union of an arbitrary number of open sets is open, each of the O_k is open. Furthermore, $O = \bigcup_{k=1}^{\infty} O_k$ is open. Now, we have that $\hat{\mathcal{C}} = [0,1] \setminus O$ is closed and therefore measurable (as all closed sets are measurable).

Now, to determine $m(\hat{\mathcal{C}})$, observe that both O and $\hat{\mathcal{C}}$ are measurable (O is measurable because it is open) and disjoint and that $O \cup \hat{\mathcal{C}} = [0,1]$. Then

$$\begin{aligned} m([0,1]) &= m(O) + m(\hat{\mathcal{C}}) \\ m(\hat{\mathcal{C}}) &= m([0,1]) - m(O) \end{aligned}$$

Furthermore observe that all of the O_k are open (and so measurable) and disjoint with $O = \bigcup_{k=1}^{\infty} O_k$. If we further break the O_k into their constituent open subsets, these properties still hold. Hence

$$\begin{aligned} m(\hat{\mathcal{C}}) =& m([0,1]) - m(O) \\ =& m([0,1]) - \sum_{k=1}^{\infty} m(O_k) \\ =& 1 - \sum_{k=1}^{\infty} 2^{k-1}l_k \end{aligned}$$

□

b) Show that if $x \in \hat{\mathcal{C}}$, then there exists a sequence of points $\{x_n\}_{n=1}^{\infty}$ such that $x_n \notin \hat{\mathcal{C}}$, yet $x_n \to x$ and $x_n \in I_n$, where I_n is a sub-interval in the complement of $\hat{\mathcal{C}}$ with $|I_n| \to 0$.

Proof. Observe first that since $\sum_{k=1}^{\infty} 2^{k-1}l_k < 1$, the tail of the series must go to zero. That is, for any $\epsilon > 0$, there exists N such that $l_n < \epsilon$ for all $n \geq N$. Now, let $x \in \hat{\mathcal{C}}$. Let $\hat{\mathcal{C}}_k$ denote the k stage of the construction. For each k, x belongs to some closed subset S_k of C_k. Let I_k be the open interval removed from S_k to proceed to the next step of the construction. We take any $x_k \in I_k$ to form our sequence $\{x_n\}_{n=1}^{\infty}$. Clearly, each x_k belongs to an sub-interval in the complement of $\hat{\mathcal{C}}$. Furthermore, $|I_k| = l_k \to 0$. It remains to show that $x_n \to x$.

From the construction of $\hat{\mathcal{C}}_k$ and our selection of x_n, it is clear that

$$|x - x_n| < |I_n| + |S_n|$$

By our previous observation, we know that $|I_n| = l_n \to 0$. Now

$$\begin{aligned} |S_n| &= \frac{1 - \sum_{k=1}^{n} 2^{k-1} l_k}{2^n} \\ &\leq \frac{1}{2^n} \\ &\to 0 \text{ as } n \to \infty \end{aligned}$$

Hence, $|x - x_n| \to 0$. That is, $\{x_n\}_{n=1}^{\infty}$ converges to x. □

c) Prove as a consequence that $\hat{\mathcal{C}}$ is perfect and contains no open interval.

Proof. To see that $\hat{\mathcal{C}}$ is perfect, let $\epsilon > 0$ be given and consider $B(x, \epsilon)$ for any $x \in \hat{\mathcal{C}}$. We can find $N \in \mathbb{N}$ such that $S_N \subset B(x, \epsilon)$. Now, this interval must have two endpoints a_N and b_N (one of which could possibly be equal to x). By the construction of $\hat{\mathcal{C}}$, we know that the endpoints of any interval are never removed, and so $a_N, b_N \in \hat{\mathcal{C}}$. Furthermore, we have that $a_N, b_N \in S_N \subset B(x, \epsilon)$. Therefore, x is not isolated.

Suppose, to the contrary, that there exists an open interval $O \in \hat{\mathcal{C}}$. Then, for any $x \in O$, there exists ϵ_0 such that $B(x, \epsilon_0) \subseteq O$. Let $\epsilon < \epsilon_0$. Then, there can be no sequence $\{x_n\}_{n=1}^{\infty}$ of the type described in part (b) whose limit is x, since $B(x, \epsilon_0) \subseteq \hat{\mathcal{C}}$ implies that $|x - x_n| > \epsilon_0 > \epsilon$ for all n. This contradicts the conclusion of part (b), and so it must be that $\hat{\mathcal{C}}$ contains no open interval. □

d) Show also that $\hat{\mathcal{C}}$ is uncountable.

Proof. We claim that $\hat{\mathcal{C}}$ is in one-to-one correspondence with infinite ternary strings containing only 0s and 2s, and so is uncountable.

($\Rightarrow$) Let $x \in \hat{\mathcal{C}}$. We build a ternary string for x of the desired form as follows. Consider $\hat{\mathcal{C}}_1$. When we remove the centrally situated open interval, it must be that x belongs to either the left closed subinterval (in which case let first digit of the ternary string for x be 0) or the right closed subinterval (in which case let first digit of the ternary string for x be 2). Next, consider $\hat{\mathcal{C}}_2$. The interval of $\hat{\mathcal{C}}_1$ to which x currently belongs will be divided into three subintervals, and so we append a 0 to the ternary string for x if it belongs to the leftmost subinterval or a 2 if it belongs to the rightmost subinterval. Continuing in this way, we see that x has an associated ternary string containing only the digits 0 and 2.

($\Leftarrow$) Let s be an infinite ternary string containing only 0s and 2s. We associate can with s an $x \in \hat{\mathcal{C}}$ as follows. If the first digit of s is 0, we choose the left subinterval of $\hat{\mathcal{C}}_1$. If the first digit of s is 2, we choose the rightmost subinterval of $\hat{\mathcal{C}}_1$. When we form $\hat{\mathcal{C}}_2$, the interval we have just chosen will be subdivided into three subintervals. If the second digit of s is 0, we select the leftmost subinterval. If the second digit of s is 2, we select the rightmost subinterval. Continue in this way. Since each $x \in \hat{\mathcal{C}}$ belongs to a singleton set, we see that s will specify some $x \in \hat{\mathcal{C}}$. □

Problem 2

Suppose E is a given set and $\mathcal{O}_n$ is the open set

$$\mathcal{O}_n = \{x : d(x, E) < \tfrac{1}{n}\}$$

a) Show that if E is compact, then $m(E) = \lim_{n \to \infty} m(\mathcal{O}_n)$.

Proof. By the Heine-Borel theorem, E is closed and bounded. Now, since E is closed, E is measurable. Now, we want to apply the following fact

$$\text{If } \mathcal{O}_k \searrow E \text{ and } m(\mathcal{O}_k) < \infty \text{ for some } k, \text{ then } m(E) = \lim_{n \to \infty} m(\mathcal{O}_n)$$

It is clear that $\mathcal{O}_k \supseteq \mathcal{O}_{k+1}$ for all k (if $d(x,E) < \frac{1}{k+1}$, then certainly $d(x,E) < \frac{1}{k}$). We show next that $E = \bigcap_{k=1}^{\infty} \mathcal{O}_k$.

Let $x \in E$. Then, $d(x,E) = 0 < \frac{1}{k}$ for all k. Hence, $x \in \mathcal{O}_k$ for all k, and therefore $x \in \bigcap_{k=1}^{\infty} \mathcal{O}_k$. That is, $E \subseteq \bigcap_{k=1}^{\infty} \mathcal{O}_k$.

Let $x \in \bigcap_{k=1}^{\infty} \mathcal{O}_k$. Then, $d(x,E) < \frac{1}{k}$ for all k. That is, $d(x,E) \to 0$. Now, since E is compact, $d(x,E)$ attains its minimum, and so $d(x,E) = 0$. This implies that $x \in E$, and therefore $\bigcap_{k=1}^{\infty} \mathcal{O}_k \subseteq E$.

To see that $m(\mathcal{O}_k) < \infty$ for some k, let N be any fixed natural number. Since E is bounded, there exists $x \in E$ and $0 < r < \infty$ such that $E \subseteq B(x,r)$. Now, let $y \in \mathcal{O}_N$. It follows that

$$\begin{aligned} d(x,y) &\le r + d(y,E) \\ &< r + \frac{1}{N} \end{aligned}$$

and so $y \in B(x, r + \frac{1}{N})$. Therefore, $\mathcal{O}_N \subseteq B(x, r + \frac{1}{N})$. Now, since $B(x, r + \frac{1}{N})$ is an open ball of finite radius, it is measurable with finite measure. Since $\mathcal{O}_N$ is open with $\mathcal{O}_N \subseteq B(x, r + \frac{1}{N})$, $\mathcal{O}_N$ is measurable with finite measure.

Having satisfied the hypotheses of the aforementioned fact, we conclude that $m(E) = \lim_{n\to\infty} m(\mathcal{O}_n)$. ☐

b) Show that the conclusion in (a) may be false for E closed and unbounded or E open and bounded.

Proof. Let $E = \mathbb{Z}$ (which is closed and unbounded) in $\mathbb{R}$. Since $\mathbb{Z}$ is a collection of singleton points, $m(\mathbb{Z}) = 0$. On the other hand, each O_n is the union of countably infinitely many open balls of radius $\frac{1}{n}$, and so $m(O_n)$ is infinite for all n. Hence, $\lim_{n\to\infty} m(O_n) = \infty$.

Let $\{r_1, \dots, r_n, \dots\} = \mathbb{Q} \cap (0,1)$. Let $E_n = (r_n - \frac{\epsilon}{2^{n+1}}, r_n + \frac{\epsilon}{2^{n+1}})$ for all n. Finally, let

$$E = (\bigcup_{n=1}^{\infty} E_n) \bigcap (0,1)$$

(which is open and bounded). Now, we have

$$O_n = \bigcup_{k=1}^{\infty} (r_k - \frac{\epsilon}{2^{k+1}} - \frac{1}{n}, r_k + \frac{\epsilon}{2^{k+1}} + \frac{1}{n})$$

Now, O_n is an open cover of $(0,1)$ for all n since, for any $x \in (0,1)$, we can find a rational number r_k within $\frac{1}{n}$ of x, and so $x \in B(r_k, \frac{1}{n}) \subseteq O_n$. Hence, $m(O_n) \ge 1$ for all n. Therefore, for $\epsilon < 1$,

$$m(E) \le \epsilon < 1 \le \lim_{n\to\infty} m(O_n)$$

☐

<u>Problem 4</u> (The Borel-Cantelli Lemma)

Suppose $\{E_k\}_{k=1}^{\infty}$ is a countable family of measurable subsets of $\mathbb{R}^d$ and that

$$\sum_{k=1}^{\infty} m(E_k) < \infty$$

Let

$$\begin{aligned} E &= \{x \in \mathbb{R}^d \mid x \in E_k, \text{ for infinitely-many } k\} \\ &= \limsup_{k\to\infty}(E_k) \end{aligned}$$

a) Show that E is measurable.

Proof. Observe first that

$$E = \bigcap_{n=1}^{\infty} \bigcup_{k\geq n} E_k$$

since, if $x \in E_k$ for infinitely many k, it will appear in the union for all k, and will be included in the countable intersection. Now, since each E_k is measurable, $\bigcup\limits_{k\geq n} E_k$ is measurable for all n (since the countable union of countable sets is countable). This implies that $E = \bigcap\limits_{n=1}^{\infty} \bigcup\limits_{k\geq n} E_k$ is countable (since the countable intersection of measurable sets is measurable). ☐

b) Prove $m(E) = 0$.

Proof. Since $\sum\limits_{k=1}^{\infty} m(E_k) < \infty$, it must be that the tail of the summation goes to 0. That is, for any $\epsilon > 0$, there exists N such that

$$\sum_{k=N}^{\infty} m(E_k) < \epsilon$$

Now,

$$E = \bigcap_{n=1}^{\infty} \bigcup_{k\geq n} E_k \subseteq \bigcup_{k=N}^{\infty} E_k$$

and so we conclude

$$\begin{aligned} \epsilon &> \sum_{k=N}^{\infty} m(E_k) \\ &\geq m(\bigcup_{k=N}^{\infty} E_k) \\ &\geq m(\bigcap_{n=1}^{\infty} \bigcup_{k\geq n} E_k) \\ &= m(E) \end{aligned}$$

and so $m(E) = 0$. ☐