Math 704 Homework 11

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April 23, 2009

Problem 1

Let $F : [0,1] \to \mathbb{R}$ such that $F'(x)$ exists almost everywhere and satisfies $F' \in L^1([0,1])$. Assume F is continuous at 0 and absolutely continuous on $[\epsilon, 1]$ for all $\epsilon > 0$. Prove that F is absolutely continuous on $[0,1]$ and thus of bounded variation on $[0,1]$.

Proof. Since F is absolutely continuous on $[\epsilon, 1]$, we have for any $\epsilon > 0$

$$F(x) = F(\epsilon) + \int_\epsilon^x F'(y)dy$$

for all $x \in [\epsilon, 1]$ (by the Second Fundamental Theorem of Calculus). Now, letting $\epsilon \to 0$

$$F(x) = F(0) + \lim_{\epsilon \to 0} \int_\epsilon^x F'(y)dy$$

for all $x \in [0,1]$ (since F is continuous at 0).

We claim that $\lim_{\epsilon \to 0} \int_\epsilon^x F'(y)dy = \int_0^x F'(y)dy$. For any $\epsilon > 0$, we have

$$\left| \int_0^x F'(y)dy - \int_\epsilon^x F'(y)dy \right| = \left| \int_0^\epsilon F'(y)dy \right|$$

which can be made arbitrarily small since $f \in L^1([0,1])$, thus proving the claim.

Therefore, we have that

$$F(x) = F(0) + \int_0^x F'(y)dy$$

for all $x \in [0,1]$, and so F is absolutely continuous on $[0,1]$ (by the Second Fundamental Theorem of Calculus). As a consequence, we have that F is of bounded variation on $[0,1]$. □

Problem 2

Let $a > b > 0$ and define $F(0) = 0$, $F(x) = x^a \sin\left(\frac{1}{x^b}\right)$ for $0 < x \leq 1$. Prove that F is of bounded variation on $[0,1]$.

Proof. Observe that

$$F'(x) = ax^{a-1} \sin\left(\frac{1}{x^b}\right) - bx^{a-b-1} \cos\left(\frac{1}{x^b}\right)$$

which is defined on $(0,1]$. That is, $F'(x)$ exists almost everywhere on $[0,1]$.

We see that $F' \in L^1([0,1])$, since

$$\begin{aligned}
\int_0^1 |F'(x)|dx &= \int_0^1 \left| ax^{a-1} \sin\left(\frac{1}{x^b}\right) - bx^{a-b-1} \cos\left(\frac{1}{x^b}\right) \right| dx \\
&\le \int_0^1 \left| ax^{a-1} \sin\left(\frac{1}{x^b}\right) \right| dx + \int_0^1 \left| bx^{a-b-1} \cos\left(\frac{1}{x^b}\right) \right| dx \\
&\le \int_0^1 \left| ax^{a-1} \right| dx + \int_0^1 \left| bx^{a-b-1} \right| dx \\
&= \int_0^1 ax^{a-1} dx + \int_0^1 bx^{a-b-1} dx \\
&\le 1 + \frac{b}{a-b} && \text{(since } a > b > 0\text{)} \\
&< \infty
\end{aligned}$$

Next, we claim that F is continuous at 0, since

$$\begin{aligned}
\lim_{x\to 0} -x^a \quad &\le \lim_{x\to 0} F(x) \quad \le \lim_{x\to 0} x^a \\
0 \quad &\le \lim_{x\to 0} F(x) \quad \le 0
\end{aligned}$$

and $F(0) = 0$ by definition.

Now, since $F'(x)$ is integrable on $[\epsilon, 1]$ for all $\epsilon > 0$, we have that $F(x) = \int_\epsilon^x F'(y)dy$ is absolutely continuous on $[\epsilon, 1]$. By problem 1, we get further than $F(x)$ is absolutely continuous on $[0,1]$, and so of bounded variation on $[0,1]$. □

Problem 3

Let $f : [0,1] \to \mathbb{R}$. Prove that the following are equivalent.

1. f is absolutely continuous, $f'(x) \in \{0,1\}$ almost everywhere, and $f(0) = 0$.
2. There exists a measurable set $A \subset [0,1]$ such that $f(x) = m(A \cap (0,x))$.

Proof. $(1 \Rightarrow 2)$ Define A to be the set $\{x \in [0,1] \mid f'(x) = 1\}$. Since f is continuous, f is measurable, so f' is measurable, which in turn gives that A is measurable. Now, since f is absolutely continuous

$$\begin{aligned}
f(x) &= f(0) + \int_0^x f'(y)dy \\
&= \int_0^x f'(y)dy \\
&= \int_0^x \chi_A(y)dy \\
&= \int_0^1 \chi_{A\cap(0,x)} dy \\
&= m(A \cap (0,x))
\end{aligned}$$

$(2 \Rightarrow 1)$ We have immediately that

$$\begin{aligned}
f(x) &= m(A \cap (0,x)) \\
&= \int_0^x \chi_A(y)dy
\end{aligned}$$

and so f is absolutely continuous (since $\chi_A \in L^1(\mathbb{R})$). By Lebesgue's Differentiation theorem, we have that $f'(x) = \chi_A(x)$ almost everywhere. Hence, $f'(x) \in \{0,1\}$ almost everywhere and $f(0) = 0$. □

Problem 4

Let f_n be absolutely continuous on $[0,1]$ and let $f_n(0) = 0$. Assume that

$$\textstyle\int_0^1 |f_n'(x) - f_m'(x)|dx \to 0$$

as $m, n \to \infty$. Prove that f_n converges uniformly to a function f on $[0,1]$ and that f is absolutely continuous on $[0,1]$.

Proof. Since $L^1(\mathbb{R})$ is a Banach space, we know that the Cauchy sequence $\{f_n'\}$ converges to some function g in norm. Let $f = \int_0^x g(y)dy$, which is absolutely continuous since $g \in L^1(\mathbb{R})$. We claim that f satisfies the remaining criteria.

Observe that, since each f_n is absolutely continuous and $f_n(0) = 0$,

$$\begin{aligned} f_n(x) &= f_n(0) + \int_0^x f_n'(y)dy \\ &= \int_0^x f_n'(y)dy \end{aligned}$$

Now

$$\begin{aligned} |f_n(x) - f(x)| &= \left|\int_0^x f_n'(y)dy - \int_0^x g(y)dy\right| \\ &= \left|\int_0^x f_n'(y) - g(y)dy\right| \\ &\leq \int_0^x |f_n'(y) - g(y)|dy \\ &\leq \int_0^1 |f_n'(y) - g(y)|dy \\ &\to 0 \end{aligned}$$

Hence, $f_n \to f$ pointwise. In fact, this convergence is uniform. Since the f_n are absolutely continuous, they are of bounded variation, and so they are bounded. Hence, for each n, there is M_n so that $|f_n - f| \leq M_n$ for all x. Since $f_n \to f$, $M_n \to 0$. So, given any $\epsilon > 0$, pick N so that $M_n < \epsilon$ for all $n \geq N$. This gives $|f_n - f| < \epsilon$ for all $n \geq N$ and for all x. That is, $f_n \to f$ uniformly. □

Problem 5

Let $f : [a,b] \to [c,d]$ be an increasing absolutely continuous function and let $g : [c,d] \to \mathbb{R}$ be an absolutely continuous function. Prove that the composition $g \circ f : [a,b] \to \mathbb{R}$ is absolutely continuous.

Proof. Let $\epsilon > 0$ be given. Since g is absolutely continuous, there exists $d > 0$ such that

$$\sum_{i=1}^{n} |g(d_i) - g(c_i)| < \epsilon$$

whenever $\{(c_i, d_i) \mid i = 1, \ldots, n\}$ are disjoint open intervals with $\displaystyle\sum_{i=1}^{n}(d_i - c_i) < \delta$. Similarly, since f is absolutely continuous, there exists a $d' > 0$ such that

$$\sum_{i=1}^{n} |f(b_i) - f(a_i)| < \delta$$

whenever $\{(a_i, b_i) \mid i = 1, \ldots, m\}$ are disjoint open intervals with $\displaystyle\sum_{i=1}^{m}(b_i - a_i) < \delta'$. Hence, for any $\{(x_i, y_i) \mid i = 1, \ldots, l\}$ disjoint open intervals with $\displaystyle\sum_{i=1}^{l}(y_i - x_i) < \delta'$, have that that $\{(f(x_i), f(y_i)) \mid i = 1, \ldots, l\}$ are disjoint open intervals (since f is increasing) with $\sum_{i=1}^{l}(f(y_i)-f(x_i)) < \delta$, and so $\sum_{i=1}^{l}|g(f(y_i))-g(f(x_i))| < \epsilon$, as desired. $\square$