Math 704 Homework 10

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Problem 1

Consider the function on $\mathbb{R}$ defined by

$$f(x) = \begin{cases} \frac{1}{|x|(ln\frac{1}{|x|})^2} & \text{if } |x| \leq \frac{1}{2} \\ 0 & \text{otherwise} \end{cases}$$

a. Verify that f is integrable.

Proof. We see that, for all $x \in \mathbb{R} \setminus \{0\}$, $f(x) \geq 0$ and f is continuous (and so measurable). We compute the integral directly.

$$\begin{aligned}
\int_{\mathbb{R}} f(x)dx &= \int_{[\frac{-1}{2},\frac{1}{2}]} \frac{1}{|x|\left(ln\frac{1}{|x|}\right)^2} dx \\
&= 2\int_{[0,\frac{1}{2}]} \frac{1}{x\left(ln\frac{1}{x}\right)^2} dx \\
&= 2\lim_{h\to 0}\left[\frac{1}{ln\frac{1}{x}}\right]_h^{\frac{1}{2}} \\
&= \frac{2}{ln(2)} \\
&< \infty
\end{aligned}$$

□

b. Establish the inequality

$$f^*(x) \geq \frac{c}{|x|ln\frac{1}{|x|}}$$

for some $c > 0$ and all $|x| \leq \frac{1}{2}$ to conclude that the maximal function f^* is not locally integrable.

Proof.

$$\begin{aligned}
f^*(x) &= \sup_B \frac{1}{m(B)}\int_B \frac{1}{|x|(ln\frac{1}{|x|})^2} \\
&\geq \frac{1}{2|x|}\int_{[-|x|,|x|]} \frac{1}{|x|(ln\frac{1}{|x|})^2} \\
&= \frac{2}{2|x|}\int_{[0,|x|]} \frac{1}{x(ln\frac{1}{x})^2} \\
&= \frac{1}{|x|ln\frac{1}{|x|}}
\end{aligned}$$

Now,

$$\begin{aligned}
\int_{[0,\frac{1}{2}]} |f^*(x)|dx &\geq \int_{[0,\frac{1}{2}]} \left| \frac{1}{|x|ln\frac{1}{|x|}} \right| dx && \text{(by above)} \\
&= \int_{[0,\frac{1}{2}]} \frac{1}{xln\frac{1}{x}} dx \\
&= \lim_{h\to 0} \left[ln\left(ln\left(\frac{1}{x} \right) \right) \right]_h^{\frac{1}{2}} \\
&= -\infty
\end{aligned}$$

Hence, f^* is not locally integrable. □

Problem 2

Consider the function $F(x) = x^2 \sin\left(\frac{1}{x^2}\right)$, $x \neq 0$ with $F(0) = 0$. Show that $F'(x)$ exists for every x, but F' is not integrable on $[-1, 1]$.

Proof. We have immediately that

$$F'(x) = 2\left(x \sin\left(\tfrac{1}{x^2}\right) - \tfrac{1}{x}\cos\left(\tfrac{1}{x^2}\right)\right)$$

which is finite everywhere except possibly at $x = 0$. We check this case separately using the definition.

$$\begin{aligned}
F'(0) &= \lim_{h\to 0} \frac{F(h) - F(0)}{h} \\
&= \lim_{h\to 0} \frac{h^2 \sin\left(\frac{1}{h^2}\right)}{h} \\
&= \lim_{h\to 0} h \sin\left(\frac{1}{h^2}\right) \\
&= 0 && \left(\text{since } \sin\left(\frac{1}{h^2}\right) \text{ is bounded}\right)
\end{aligned}$$

To show that $F'(x)$ is not integrable on $[-1, 1]$, it is sufficient to show that $\frac{1}{x}\cos\frac{1}{x^2}$ is not integrable over $[0, 1]$. We accomplish this by approximating the area under the function by triangles.

$$\begin{aligned}
\int_{[0,1]} \frac{1}{x}\cos\frac{1}{x^2}dx &\geq \frac{1}{2}\sum_{k=1}^{\infty} \left((\frac{\pi}{2} + (k+1)\pi)^{\frac{-1}{2}} - (\frac{\pi}{2} + k\pi)^{\frac{-1}{2}} \right) (k\pi)^{\frac{1}{2}} \\
&= \frac{1}{2}\sum_{k=1}^{\infty} \frac{\sqrt{k}}{\sqrt{(\frac{3}{2}+k)(\frac{1}{2}+k)} \cdot (\sqrt{\frac{3}{2}+k} + \sqrt{\frac{1}{2}+k})}
\end{aligned}$$

Applying the limit comparison test (using $\frac{1}{k}$), we have

$$\begin{aligned}
\lim_{k\to\infty} \frac{k^{\frac{3}{2}}}{\sqrt{(\frac{3}{2}+k)(\frac{1}{2}+k)} \cdot (\sqrt{\frac{3}{2}+k} + \sqrt{\frac{1}{2}+k})} &= \lim_{k\to\infty} \frac{1}{\sqrt{1+\frac{2}{k}+\frac{3}{4k^2}} \cdot (\sqrt{\frac{3}{2k}+1} + \sqrt{\frac{1}{2k}+1})} \\
&= \frac{1}{2}
\end{aligned}$$

Since $\displaystyle\sum_{k=1}^{\infty}\frac{1}{k}$ diverges, the limit comparison test implies that our original sum diverges, as well. Therefore, $\frac{1}{x}\cos\frac{1}{x^2}$ is not integrable over $[0, 1]$, completing the proof. □

Problem 3

Suppose F is of bounded variation and continuous. Prove that $F = F_1 - F_2$, where both F_1 and F_2 are monotonic and continuous.

Proof. Let $[a, b]$ with $a < b$ be any closed interval in $\mathbb{R}$. For $x \in [a, b]$, we have

$$\begin{aligned} F(x) - F(a) &= P_a^x F - N_a^x F \\ F(x) &= (P_a^x F + F(a)) - N_a^x F. \end{aligned}$$

Identifying F_1 with $P_a^x F + F(a)$ and F_2 with $N_a^x F$, it suffices to show that both $P_a^x F$ and $N_a^x F$ are continuous on $[a, b]$ (as they are obviously monotonic). Since

$$T_a^x F = P_a^x F - N_a^x F,$$

it further suffices to show that $T_a^x F$ is continuous on $[a, b]$. To see this, let $\tilde{x} \in [a, b]$ and let $\epsilon > 0$ be given. Since F is uniformly continuous over the compact set $[a, b]$, we can find $\delta > 0$ small enough to ensure that $|F(x) - F(y)| \leq \frac{\epsilon}{3}$ whenever $|x - y| < \delta$ for all $x, y \in [a, b]$. Now, choose a partition P of $[a, b]$ such that the distance between any two consecutive elements of the partition is less than δ and

$$T_a^b F < \sum_{k=1}^{N} |F(x_k) - F(x_{k-1})| + \frac{\epsilon}{3}$$

where the x_k are the elements of P. Without loss of generality, we may assume that one of these elements, say x_l, is our $\tilde{x}$, since a refinement will only increase the precision of the estimate. Now, by restricting our view to the interval $[x_{l-1}, x_{l+1}]$, we have

$$\begin{aligned} T_{x_{l-1}}^{x_{l+1}} F &< |F(x_l) - F(x_{l-1})| + |F(x_{l+1}) - F(x_l)| + \frac{\epsilon}{3} \\ &< \frac{\epsilon}{3} + \frac{\epsilon}{3} + \frac{\epsilon}{3} && \text{(since } F \text{ continuous at } \tilde{x} \text{ and } x\text{'s are sufficiently close)} \\ &= \epsilon \end{aligned}$$

Hence, the function T_a^x is continuous on any interval $[a, b]$, thus proving the original claim. $\square$

Problem 4

a. Let $F : [a, b] \to \mathbb{R}$ be a function of bounded variation and let $a < c < b$. Prove that $T_a^c F + T_c^b F = T_a^b F$. (Here, $T_a^b F$ denotes the total variation of F on $[a, b]$.)

Proof. For any partition P, denote its elements by x_k.

$$\begin{aligned} T_a^b F &= \sup_P \sum_{x_k \in [a,b]} |f(x_k) - f(x_{k-1})| \\ &= \sup_{P \cup \{c\}} \sum_{x_k \in [a,b]} |f(x_k) - f(x_{k-1})| && \text{(inclusion of the point } c \text{ can only refine } P) \\ &= \sup_{P \cup \{c\}} \left\{ \sum_{x_k \in [a,c]} |f(x_k) - f(x_{k-1})| + \sum_{x_k \in [c,b]} |f(x_k) - f(x_{k-1})| \right\} \\ &= \sup_{P \cup \{c\}} \sum_{x_k \in [a,c]} |f(x_k) - f(x_{k-1})| + \sup_{P \cup \{c\}} \sum_{x_k \in [c,b]} |f(x_k) - f(x_{k-1})| && \text{(since } [a,c] \text{ and } [c,b] \text{ are almost disjoint)} \\ &= T_a^c F + T_c^b F \end{aligned}$$

$\square$

b. Let F be as in part (a). Prove that

$$\int_a^b |F'(x)| dx \leq T_a^b F.$$

Proof. Observe first

$$\begin{aligned} T_x^{x+h}F &= \sup_P \sum_{x_k \in [x,x+h]} |F(x_k) - F(x_{k-1})| \\ &\geq |F(x+h) - F(x)| \qquad \text{(this is the sum over a \textit{particular} partition of } [x, x+h]) \end{aligned}$$

It follows

$$\begin{aligned} |F'(x)| &= \lim_{h\to 0} \left| \frac{F(x+h) - F(x)}{h} \right| \\ &= \lim_{h\to 0} \frac{|F(x+h) - F(x)|}{h} \\ &\leq \lim_{h\to 0} \frac{T_x^{x+h}}{h} && \text{(by above)} \\ &= \lim_{h\to 0} \frac{T_a^{x+h} - T_a^x}{h} && \text{(by part (a))} \\ &= (T_a^x F)' \end{aligned}$$

Finally, we have

$$\begin{aligned} \int_a^b |F'(x)| dx &\leq \int_a^b (T_a^x F)' dx && \text{(by above)} \\ &= T_a^b F - T_a^a F \\ &= T_a^b F \end{aligned}$$

□

<u>Problem 5</u>

Let $a \leq b$ and define $F(0) = 0$, $F(x) = x^a \sin \frac{1}{x^b}$ for $0 < x \leq 1$. Prove that F is not of bounded variation on $[0, 1]$.

Proof. Let $x_k = \left(\frac{\pi}{2} + k\pi\right)^{\frac{-1}{b}}$. Observe that, when k is even, $\sin \frac{1}{x_k^b} = 1$, and when k is odd, $\sin \frac{1}{x_k^b} = -1$. Now, for any finite sum of the x_k,

$$\begin{aligned} \sum_{k=1}^{N} |f(x_k) - f(x_{k-1})| &= \sum_{k=1}^{N} |(-1)^k (x_k^a + x_{k-1}^a)| \\ &= \sum_{k=1}^{N} (x_k^a + x_{k-1}^a) \\ &= x_N + x_0 + 2 \sum_{k=1}^{N-1} x_k^a \\ &\geq \sum_{k=1}^{N-1} x_k^a \\ &= \pi^{\frac{-a}{b}} \sum_{k=1}^{N-1} \left(\frac{1}{2} + k\right)^{\frac{-a}{b}} \end{aligned}$$

which diverges as $N \to \infty$ (since $a \leq b$). Therefore, F is not of bounded variation on $[0, 1]$. □