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Math 704
Homework 1

Problem 1

Prove that the Cantor set $\mathcal{C}$ is totally disconnected and perfect. In other words, given two distinct points $x, y \in \mathcal{C}$, there is a point $z \notin \mathcal{C}$ that lies between x and y, and yet $\mathcal{C}$ has no isolated points.

Proof. Let $x, y \in \mathcal{C}$ be distinct. Then, $x, y \in \mathcal{C}_k$ for all $k \in \mathbb{N}$. Now, since x and y are distinct, we can find $N \in \mathbb{N}$ such that $\frac{1}{3^N} < |x - y|$. Hence, x and y belong to different intervals of $\mathcal{C}_N$. By the construction of the Cantor set, there must be at least one interval between x and y which does not belong to $\mathcal{C}_N$, and so does not belong to $\mathcal{C}$. Select one such interval. Choosing any point z in this interval satisfies that z lies between x and y and $z \notin \mathcal{C}$. Therefore, $\mathcal{C}$ is totally disconnected.

To see that $\mathcal{C}$ is perfect, let $\epsilon > 0$ be given and consider $B(x, \epsilon)$ for any $x \in \mathcal{C}$. Let I_k denote the interval to which x belongs in $\mathcal{C}_k$. We can find $N \in \mathbb{N}$ such that $I_N \subset B(x, \epsilon)$. Now, this interval must have two endpoints a_N and b_N (one of which could possibly be equal to x). By the construction of the Cantor set, we know that the endpoints of any interval are never removed, and so $a_N, b_N \in \mathcal{C}$. Furthermore, we have that $a_N, b_N \in I_N \subset B(x, \epsilon)$. Therefore, x is not isolated. $\square$

Problem 2

The Cantor set $\mathcal{C}$ can also be described in terms of ternary expansions.

a) Every number in $[0, 1]$ has a ternary expansion

$$x = \sum_{k=1}^{\infty} \frac{a_k}{3^k}, \text{ where } a_k = 0, 1, \text{ or } 2.$$

Prove that $x \in \mathcal{C}$ if and only if x has a representation as above where every a_k is either 0 or 2.

Proof. ($\Rightarrow$) Let $x \in \mathcal{C}$. We build a ternary expansion for x of the desired form as follows.Consider $\mathcal{C}_1$. It must be that x belongs to one of $[0, \frac{1}{3}]$ (in which case let first digit of the ternary expansion for x be 0) or $[\frac{2}{3}, 1]$ (in which case let first digit of the ternary expansion for x be 2). Next, consider $\mathcal{C}_2$. The interval of $\mathcal{C}_1$ to which x currently belongs will be divided into three subintervals, and so we append a 0 to the ternary expansion of x if it belongs to the leftmost subinterval or a 2 if it belongs to the rightmost subinterval. Continuing in this way, we see that x has an associated ternary expansion containing only the digits 0 and 2.

($\Leftarrow$) Let

$$x = \sum_{k=1}^{\infty} \frac{a_k}{3^k}, \text{ where } a_k = 0 \text{ or } 2.$$

We can locate x on the real line as follows. If $a_1 = 0$, we choose the left subinterval of $\mathcal{C}_1$. If $a_1 = 2$, we choose the rightmost subinterval of $\mathcal{C}_1$. When we form $\mathcal{C}_2$, the interval we have just chosen will be subdivided into three subintervals. If $a_2 = 0$, we select the leftmost subinterval. If $a_2 = 2$, we select the rightmost subinterval. Continue in this way. Since the length of these intervals can be made arbitrarily small, we see that the ternary expansion of x uniquely specifies its location on the real line. $\square$

b) The Cantor-Lebesgue function is defined on $\mathcal{C}$ by

$$F(x) = \sum_{k-1}^{\infty} \frac{b_k}{2^k} \text{ if } x = \sum_{k=1}^{\infty} \frac{a_k}{3^k}, \text{ where } b_k = \frac{a_k}{2}$$

In this definition, we choose the expansion of x in which $a_k = 0$ or 2. Show that F is well-defined and continuous on $\mathcal{C}$, and moreover $F(0) = 0$ as well as $F(1) = 1$.

Proof. Let $x, x' \in \mathcal{C}$ with $x = x'$. Denote the kth digit of the ternery expansion of x and x' by a_k and a'_k, respectively.

Claim $a_k = a'_k$ for all k.
Proof of Claim Suppose not. Then, $a_N \neq a'_N$ for some N. From the construction in part (a), we see that x and x' must belong to different subintervals in $\mathcal{C}_N$, and so $x \neq x'$, which is a contradiction.

Now, let $b_k = \frac{a_k}{2}$ and $b'_k = \frac{a'_k}{2}$. Then $b_k = b'_k$ for all k. Hence

$$F(x) = \sum_{k=1}^{\infty} \frac{b_k}{2^k} = \sum_{k=1}^{\infty} \frac{b'_k}{2^k} = F(x')$$

and so F is well-defined.

To see that F is continuous, let $\epsilon > 0$ be given and $x, x' \in \mathcal{C}$ so that $|F(x) - F(x')| < \epsilon$. Consider the binary expansion of ϵ (denote the kth digit of ϵ by ϵ_k). Construct $\delta > 0$ such that $\delta_k = 2\epsilon_k$ for all k. Let N be the first nonzero digit of δ and ϵ. Then, $|x - x'| < \delta$ implies that the first $N-1$ digits of x and x' agree. Hence, the first $N-1$ digits of $F(x)$ and $F(x')$ agree, and so $|F(x) - F(x)'| < \epsilon$. Therefore, F is continuous.

By the construction in part (a), we know that 0 is represented in ternary form by always choosing the leftmost subinterval, and so for $x = 0$, $b_k = \frac{0}{2} = 0$ for all k. Similarly, 1 is represented in ternary form by always chosing the rightmost subinterval, and so for $x = 1$, $b_k = \frac{2}{2} = 1$ for all k. Hence

$$F(0) = \sum_{k=1}^{\infty} \frac{0}{2^k} = 0$$

$$F(1) = \sum_{k=1}^{\infty} \frac{1}{2^k} = \frac{1}{2}\sum_{k=0}^{\infty} \frac{1}{2^k} = \frac{\frac{1}{2}}{1 - \frac{1}{2}} = 1$$

□

c) Prove that $F : \mathcal{C} \to [0,1]$ is surjective.

Proof. Let $y \in [0,1]$. Then y has a corresponding binary expansion. Let b_k denote the kth digit of this expansion. Construct a string s such that $s_k = 2b_k$ for all k, where s_k denotes the kth digit of s. This construction uniquely identifies some ternary string using only 0s and 2s. From part (a), we know that s corresponds uniquely to some $x \in \mathcal{C}$. Now, it is clear from our construction of x that $F(x) = y$. □

Problem 3

Recall that every open set in $\mathbb{R}$ is the disjoint union of open intervals. The analogue in $\mathbb{R}^d$, $d \geq 2$ is generally false. Prove the following:

a) An open disc in $\mathbb{R}^2$ is not the disjoint union of open rectangles.

Proof. Suppose, to the contrary, an open disc $\mathcal{O} \subset \mathbb{R}^2$ is the disjoint union of open rectangles. Choose some open rectangle $R_1 \in \mathcal{O}$ and let $x \in \delta R_1$. Then, for all $\epsilon > 0$, $B(x, \epsilon) \cap R_1 \neq \phi$ and $B(x, \epsilon) \cap R_1^c \neq \phi$. Hence, $x \notin R_1$, and so there must be an open rectangle $R_2 \in \mathcal{O}$ with $x \in R_2$. This implies that there is $\epsilon_0 > 0$ such that $B(x, \epsilon_0) \subset R_2$. By our previous observation, $B(x, \epsilon_0) \cap R_1 \neq \phi$. Taken together, we see that $R_1 \cap R_2 \neq \phi$, which is a contradiction with the fact that $\mathcal{O}$ is the disjoint union of open rectangles. □

b) An open connected set Ω is the disjoint union of open rectangles if and only if Ω is itself an open rectangle.

Proof. ($\Rightarrow$) Let Ω be the disjoint union of open rectangles. Suppose, to the contrary, that Ω is not itself an open rectangle. Then, Ω contains at least two open rectangles. By the argument in part (a), we see that these rectangles cannot be disjoint, which is a contradiction. Hence, it must be that Ω is itself an open rectangle.

($\Leftarrow$) Let Ω be an open rectangle. Then Ω is the disjoint union of a single open rectangle (namely, Ω itself). $\square$