Austin Mohr
Math 701
Homework 4

Problem 13

Let p be the smallest prime that divides the cardinality of the finite group G. Prove that any subgroup of G of index p must be normal.

Proof. Let H be a subgroup of G having index p in G. Let π_H be the action of left multiplication by G on the cosets of H in G. Since pi_H is a homomorphism, $ker(\pi_H)$ is normal in G and $[G : K] = [G : H][H : ker(\pi_H)]$. Now, since H has p left cosets, $G/_{ker(\pi_H)}$ is isomorphic to some subgroup of of SYM_p. By Lagrange's theorem, we have that $|G/_{ker(\pi_H)}| = p[H : ker(\pi_H)]$ divides $p!$. That is, $[H : ker(\pi_H)]$ divides $\frac{p!}{p} = (p-1)!$. The minimality of p implies that $[H : ker(\pi_H)] = 1$, and so $ker(\pi_H) = H$. Therefore, H is normal in G. □

Problem 14

How many elements of order 7 are there in a simple group of order 168?

Proof. Note first that $168 = 2^3 * 3 * 7$. By Sylow's theorem, we have that $|Syl^7(G)| \equiv 1 \pmod 7$ and $|Syl^7(G)| \mid 24$. So, $|Syl^7(G)|$ is equal to 1 or 8. Since G is simple, it cannot be that $|Syl^7(G)| = 1$, as this would imply that G has a unique (and so normal) Sylow 7-subgroup. Now, all elements of order 7 appear in a Sylow 7-subgroup. Furthermore, the 8 Sylow 7-subgroups are disjoint except for the identity element. Therefore, there are $8 * 6 = 48$ elements of order 7. □

Problem 15

Let N be a normal subgroup of the finite group G and let K be a Sylow p-subgroup of N for some prime p. Prove that $G = N_G(K)N$.

Proof. Let g be an element of G. Since K is a Sylow p-subgroup of the normal subgroup N, we have that gKg^{-1} is also a Sylow p-subgroup of N. By Sylow's theorem, all Sylow p-subgroups are conjugate, so we can find an element n of N so that $ngKg^{-1}n^{-1} = K$. The lefthand side can be rewritten as $(ng)K(ng)^{-1}$, and so we see that ng is an element of $N_G(K)$. That is, $g = n^{-1}(ng) \in NN_G(K)$. Since N is normal in G, $NN_G(K) = N_G(K)N$, and so we have shown that $g \in N_G(K)N$.

The reverse inclusion is easy, since $N_G(K) \leq G$ and $N \leq G$. Therefore, $G = N_G(K)N$. □

Problem 16

Prove that there is no simple group G of order 56.

Proof. Note first that $56 = 2^3 * 7$. By Sylow's theorem, we have that $|Syl^7(G)| \equiv 1 \pmod 7$ and $|Syl^7(G)| \mid 8$. So, $|Syl^7(G)|$ is equal to 1 or 8. If $|Syl^7(G)| = 1$, then G has a unique (and so normal) Sylow 7-subgroup. If $|Syl^7(G)| = 8$, then there are $8 * 6 = 48$ elements of order 7, leaving 8 elements which comprise a unique (and so normal) Sylow 2-subgroup. In either case, we see that G contains a normal subgroup, and so we see that G cannot be normal. □