Austin Mohr
Math 701
Homework 3

Problem 10

Let **G** be a group. Prove that **G** cannot have four distinct proper normal subgroups $\mathbf{N}_0, \mathbf{N}_1, \mathbf{N}_2, \mathbf{N}_3$ so that $\mathbf{N}_0 \leq \mathbf{N}_1 \leq \mathbf{N}_2 \leq \mathbf{G}$ and so that $N_1N_3 = G$ and $N_2 \cap N_3 = N_0$.

Proof. Suppose we have normal subgroups of the desired form. Let $n_2 \in N_2 \subset G$. Then, $n_2 = n_1n_3$ for some $n_1 \in N_1$ and $n_3 \in N_3$ (since $N_1N_3 = G$). We have that $n_3 = n_1^{-1}n_2 \in N_2$ (since $n_1^{-1} \in N_1 \subset N_2$). Hence, $n_3 \in N_2 \cap N_3 = N_0 \subset N_1$. Since $n_3 \in N_1$, we have that $n_2 = n_1n_3 \in N_1$. Therefore, $N_2 \subset N_1$, which implies that $N_1 = N_2$. This is a contradiction with the fact that N_1 and N_2 are distint. ☐

Problem 11

Let **H** and **K** be subgroups of the group **G** each of finite index in **G**. Prove that $\mathbf{H} \cap \mathbf{K}$ is also a subgroup of finite index in **G**.

Claim 1. *$H \cap K$ is a subset of G.*

Proof. Let $a \in H \cap K$. Then, $a \in H$. Since **H** is a subgroup of G, we have that $a \in G$. ☐

Claim 2. *$\boldsymbol{H} \cap \boldsymbol{K}$ contains the identity.*

Proof. **H** and **K** are both groups, so we have $1 \in H$ and $1 \in K$. Therefore, $1 \in H \cap K$. ☐

Claim 3. *$\boldsymbol{H} \cap \boldsymbol{K}$ is closed under the taking of inverses.*

Proof. Let $a \in H \cap K$. Then, $a \in H$ and $a \in K$. Since **H** and **K** are both groups, we have that $a^{-1} \in H$ and $a^{-1} \in K$. Therefore, $a^{-1} \in H \cap K$. ☐

Claim 4. *$\boldsymbol{H} \cap \boldsymbol{K}$ is closed under addition.*

Proof. Let $a, b \in H \cap K$. Then, a, b belong to both H and K. Since **H** and **K** are both groups, we have that $a + b$ belongs to both H and K. Therefore, $a + b \in H \cap K$. ☐

By the previous claims, we see that $\mathbf{H} \cap \mathbf{K}$ is a subgroup of **G**.

Claim 5. *$\boldsymbol{H} \cap \boldsymbol{K}$ has finite index in $\boldsymbol{G}$.*

Proof. Define the function

$$f : G/_{H \cap K} \to G/_H \times G/_K$$
$$f(a(H \cap K)) = (aH, aK) \text{ for all } a \in G$$

To see that this function is well-defined, suppose $a(H \cap K) = b(H \cap K)$. Then, a and b belong to the same equivalence class in $G/_{H \cap K}$. We see that

$$\begin{aligned} &a(H \cap K) \subseteq aH \\ \Rightarrow &b(H \cap K) \subseteq aH \\ \Rightarrow &aH = bH \end{aligned}$$

and similarly

$$\begin{aligned} &a(H \cap K) \subseteq aK \\ \Rightarrow &b(H \cap K) \subseteq aK \\ \Rightarrow &aK = bK \end{aligned}$$

Hence, $(aH, aK) = (bH, bK)$, and so f is well-defined.

We claim that this function is one-to-one. Suppose $(aH, aK) = (bH, bK)$. Then $aH = bH$ and $aK = bK$, which implies that a and b belong to the same equivalence class in ${}^{G}\!/_{H}$ as well as the same equivalence class in ${}^{G}\!/_{K}$. Therefore, a and b belong to the same equivalence class in ${}^{G}\!/_{H \cap K}$. That is, $a(H \cap K) = b(H \cap K)$.

Now, f is a one-to-one map sending each coset of ${}^{G}\!/_{H \cap K}$ into a finite range (since each of H and K have finite index in G), so it must be that the domain is finite. That is, $H \cap K$ has finite index in G. $\square$

Problem 12

Prove that there is no group $\mathbf{G}$ such that ${}^{\mathbf{G}}\!/_{\mathbf{Z(G)}} \cong \mathbb{Z}$, where $\mathbb{Z}$ denotes the group of integers under addition.

Proof. Suppose ${}^{G}\!/_{Z(G)} \cong \mathbb{Z}$ for some group G. Since $\mathbb{Z}$ is cyclic, ${}^{G}\!/_{Z(G)}$ is cyclic. Let $aZ(G)$ generate ${}^{G}\!/_{Z(G)}$. Then, every coset is of the form $a^n Z(G)$ for some integer n. Since the cosets partition G, we have that every element of G is of the form $a^n c$ for some integer n and some element $c \in Z(G)$. Let g_1, g_2 be arbitrary elements of G with $g_1 = a^{n_1} c_1$ and $g_2 = a^{n_2} c_2$. It follows that

$$\begin{aligned} g_1 g_2 &= a^{n_1} c_1 a^{n_2} c_2 \\ &= a^{n_2} c_2 a^{n_1} c_1 && \text{(since } c_1, c_2 \in Z(G) \text{ and powers of } a \text{ commute)} \\ &= g_2 g_1 \end{aligned}$$

Hence, any two elements of G commute. That is, $Z(G) = G$, and so ${}^{G}\!/_{Z(G)} = \{1\}$, which contradicts our assumption that ${}^{G}\!/_{Z(G)} \cong \mathbb{Z}$. $\square$