Austin Mohr
Math 701
Homework 2

Problem 6

Derive a list of equations that follow from the equations axiomatizing the theory of groups.

Claim 1. *The distinguished element 1 in a group is the unique identity.*

Proof. Suppose that there exists another identity element $1'$. Then

$$1 = 1 \cdot 1' = 1'$$

□

Claim 2. *The element x^{-1} is the unique inverse of the element x.*

Proof. Suppose that x has another inverse x'. Then

$$x' = x' \cdot 1 = x' \cdot (xx^{-1}) = (x'x)x^{-1} = 1 \cdot x^{-1} = x^{-1}$$

□

Claim 3. $(xy)^{-1} = y^{-1}x^{-1}$

Proof. $(xy)(y^{-1}x^{-1}) = x(yy^{-1})x^{-1} = x \cdot 1 \cdot x^{-1} = xx^{-1} = 1$ □

Claim 4. $x^n x^k = x^{n+k}$

Proof. $x^n x^k = \underbrace{(x \cdots x)}_{\text{n times}} \underbrace{(x \cdots x)}_{\text{k times}} = \underbrace{x \cdots x}_{\text{n + k times}} = x^{n+k}$ □

Claim 5. $(x^n)^{-1} = (x^{-1})^n$

Proof.

$$\begin{aligned} x^n(x^{-1})^n &= \underbrace{(x \cdots x)}_{\text{n times}} \underbrace{(x^{-1} \cdots x^{-1})}_{\text{n times}} \\ &= \underbrace{(x \cdots x)}_{\text{n-1 times}} (xx^{-1}) \underbrace{(x^{-1} \cdots x^{-1})}_{\text{n-1 times}} \\ &= \underbrace{(x \cdots x)}_{\text{n-1 times}} \cdot 1 \cdot \underbrace{(x^{-1} \cdots x^{-1})}_{\text{n-1 times}} \\ &= \underbrace{(x \cdots x)}_{\text{n-1 times}} \underbrace{(x^{-1} \cdots x^{-1})}_{\text{n-1 times}} \\ &= \cdots \\ &= xx^{-1} \\ &= 1 \end{aligned}$$

□

Problem 7

The five equations used to axiomatize groups are not all needed. Find a simpler set of equations that will serve.

We take only the following equations for our axiomatization of groups. For all elements x, y, z in the algebra $\mathbf{A} = \langle A, \cdot, {}^{-1}, 1 \rangle$

- $x(yz) = (xy)z$

- $xx^{-1} = 1$
- $x \cdot 1 = x$

From these, we derive the following.

Claim 6. $x^{-1}x = 1$

Proof.

$$\begin{aligned} x^{-1}x &= (x^{-1} \cdot 1)x \\ &= (x^{-1}(xx^{-1}))x \\ &= (x^{-1}x)(x^{-1}x) \\ (x^{-1}x)(x^{-1}x)^{-1} &= (x^{-1}x)(x^{-1}x)(x^{-1}x)^{-1} \\ 1 &= x^{-1}x \end{aligned}$$

□

Claim 7. $1 \cdot x = x$

Proof.

$$\begin{aligned} 1 \cdot x &= (xx^{-1})x \\ &= x(x^{-1}x) \\ &= x \cdot 1 \\ &= x \end{aligned}$$

□

Problem 8

State and prove a version of the isomorphism theorem from class that holds for algebraic systems generally, not just for groups.

Theorem 1. *(The Second Isomorphism Theorem) Let $\boldsymbol{A}$ be an algebra and let Θ and Φ be congruence relations on $\boldsymbol{A}$ such that $\Theta \subseteq \Phi$. Then Φ/Θ is a congruence relation on $\boldsymbol{A}/\Theta$ and*

$$\boldsymbol{A}/\Theta / \Phi/\Theta \cong \boldsymbol{A}/\Phi$$

Proof. Define $h : \mathbf{A}/\Theta \to \mathbf{A}/\Phi$ by

$$h(a/\Theta) = a/\Phi \text{ for all } a \in \mathbf{A}$$

Claim 8. *h is well-defined.*

Proof. Suppose $a/\Theta = b/\Theta$. Then $(a, b) \in \Theta \subseteq \Phi$, and so $a/\Phi = b/\Phi$. □

Claim 9. *h is onto $\boldsymbol{A}/\Phi$.*

Proof. Let $a/\Phi \in \mathbf{A}/\Phi$. We see that $h(a/\Theta) = a/\Phi$. □

Claim 10. *h is a homomorphism.*

Proof.

$$\begin{aligned} h(a/_{\Theta} \cdot b/_{\Theta}) &= h(ab/_{\Theta}) \\ &= ab/_{\Phi} \\ &= a/_{\Phi} \cdot b/_{\Phi} \\ &= h(a/_{\Theta}) \cdot h(b/_{\Theta}) \end{aligned}$$

□

Claim 11. $Ker(h) = \Phi/_{\Theta}$

Proof.

$$\begin{aligned} (a/_{\Theta}, b/_{\Theta}) \in Ker(h) &\Leftrightarrow h(a/_{\Theta}) = h(b/_{\Theta}) \\ &\Leftrightarrow a/_{\Phi} = b/_{\Phi} \\ &\Leftrightarrow (a, b) \in \Phi \end{aligned}$$

Hence

$$\begin{aligned} Ker(h) &= \{(a/_{\Theta}, b/_{\Theta}) \mid (a, b) \in \Phi\} \\ &= \Phi/_{\Theta} \end{aligned}$$

□

Now, by the Homomorphism Theorem, there is an isomorphism between $\mathbf{A}/_{\Theta}/_{\Phi/_{\Theta}}$ and $\mathbf{A}/_{\Phi}$, that is $\mathbf{A}/_{\Theta}/_{\Phi/_{\Theta}} \cong \mathbf{A}/_{\Phi}$. □