Order from Chaos

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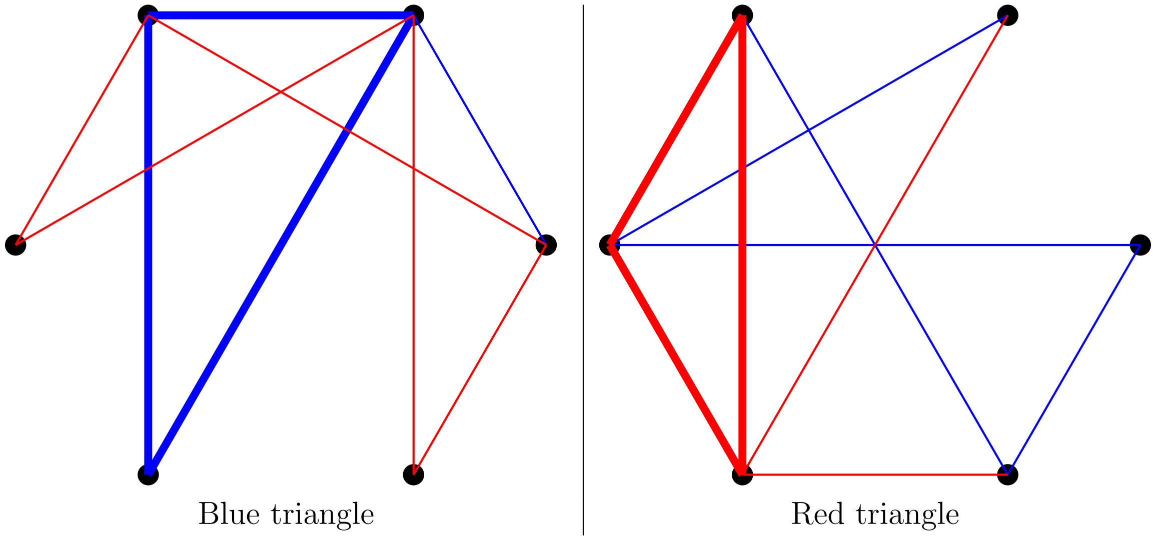
The $(3, 3)$-Puzzle

- Start by drawing six dots at the corners of a hexagon.
- One player takes a blue pen. The other player takes a red pen.
- At any time, a player may connect any two unconnected dots with their pen. *Players need not take turns.*
- If three dots ever form a triangle of a single color, both players immediately lose.
- If all fifteen possible connections are drawn without forming any triangles, both players win.



Blue triangle

Red triangle

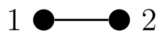
Work with your partner to find a solution to this puzzle.

Please wait for the moderator before proceeding.

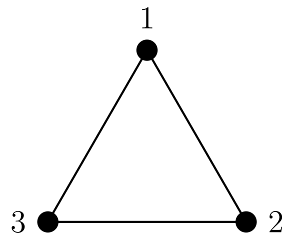
Triangles Are Unavoidable

You may be getting the feeling that the (3, 3)-Puzzle is unsolvable. You currently face a common mathematical dilemma: Do we keep searching for a solution, or do we try to prove that no solution exists? We've already spent a bit of time with the former, so let's give the latter a try.

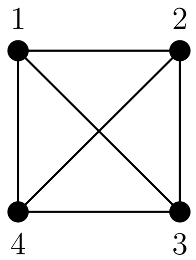
Mathematicians call the dots in our pictures **vertices** and the connections between them **edges**. The object of our current interest is called the **complete graph** on six vertices (often abbreviated K_6). It is "complete" in the sense that every pair of vertices is connected by an edge. Here are a bunch of other complete graphs. Notice that a triangle is really just a K_3.



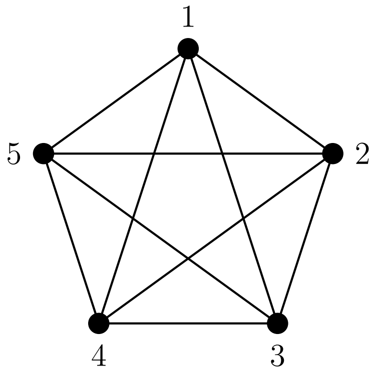
K_2



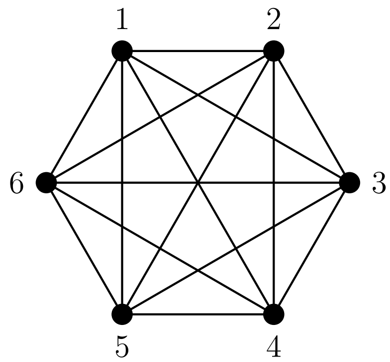
K_3



K_4



K_5



K_6

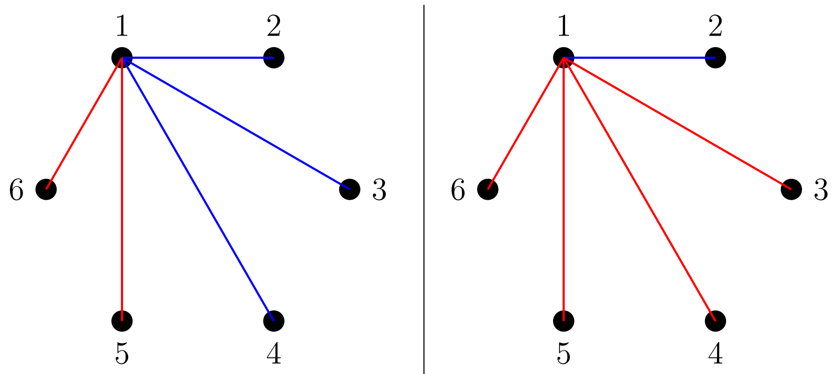
We should clarify a couple things about graphs.

- The way we are using the word "graph" in this context has nothing to do with graphs of functions from algebra and calculus.
- There is no geometric significance to the way a graph is drawn on paper. All that a graph knows is whether two vertices are connected by an edge or not. In particular, an edge always connects *exactly two* vertices (even if you might draw it in a funny way that appears to connect fewer or more vertices).

Our guess that the (3, 3)-Puzzle has no solution on six vertices can be expressed in the language of graphs.

Conjecture. *If one colors every edge of K_6 either red or blue, there will always be three vertices that form a monochromatic K_3 (that is, a triangle that is all red or all blue).*

Here is a crucial observation: Since vertex 1 has five edges connected to it, one of the colors forms a strict majority of *at least* three edges. In the pictures below, we consider examples of majorities at vertex 1. There are still many more edges to be added to form a complete graph (all the edges between vertices 2, 3, 4, 5, and 6).



Example of blue majority | Example of red majority

1. Consider the blue majority in the example. There are still lots of edges to add between vertices 2, 3, and 4. What structure is formed if even one these edges is blue?

2. The alternative in this example blue majority is that *every* edge between vertices 2, 3, and 4 is red. What structure is formed in this case?

3. Are your conclusions specific to this example, or do they always apply when there is a blue majority?

4. Can you construct a similar argument when there is a red majority?

According to this line of reasoning, *every* coloration of K_6 contains a monochromatic triangle. In other words, the (3, 3)-Puzzle has no solution on six vertices. This fact has an interesting consequence on social networks.

5. Invite any six people in the world to a party. No matter who is invited, we can always find three of them who are mutual acquaintances (i.e., all three have met each other before) or three of them who are mutual strangers (i.e., none of these three people have ever met before). How can we justify this phenomenon?

It is natural to wonder whether there is anything special about the number six.

6. Is there a solution to the (3, 3)-Puzzle on seven vertices? What about even more vertices?

7. Is there a solution to the (3, 3)-Puzzle on five vertices? What about even fewer vertices?

Please wait for the moderator before proceeding.

Ramsey Numbers

We have proven two interesting facts about the $(3, 3)$-Puzzle.

- If there are six or more vertices, then the $(3, 3)$-Puzzle has no solution.
- If there are five or fewer vertices, then the $(3, 3)$-Puzzle has a solution.

In other words, six is the *smallest* number of vertices for which the $(3, 3)$-Puzzle is unsolvable. Mathematicians condense this fact even further by writing $R(3, 3) = 6$. (The lefthand side is typically read aloud as "R three three".)

- The "R" stands for Frank Ramsey, the economist who first thought about the puzzle. It is in his honor that the numbers we are investigating are called **Ramsey numbers**.
- The first "3" indicates that the players can lose by drawing a blue K_3 (that is, a triangle).
- The second "3" indicates that the players can also lose by drawing a red K_3.

With this notation, we can extend the puzzle to other kinds of blue or red graphs.

The (m, n)-Puzzle (for any whole numbers m and n)

- Start by drawing several vertices on the circumference of a circle. You can draw as many as you like. Each number of vertices is a different version of the puzzle.
- One player takes a blue pen. The other player takes a red pen.
- At any time, a player may connect any two unconnected dots with their pen. *Players need not take turns.*
- If m dots ever form a blue K_m, both players immediately lose.
- If n dots ever form a red K_n, both players immediately lose.
- If all possible connections are drawn without forming either of these two graphs, both players win.

Our experience with the $(3, 3)$-Puzzle tells us that the (m, n)-Puzzle may become unsolvable if too many vertices are drawn, so we make the following definition.

Definition. *Let m and n be any whole numbers. The **Ramsey number** $R(m, n)$ is the* smallest *number of vertices for which the (m, n)-Puzzle is unsolvable.*

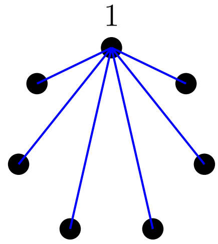
To rephrase this in terms of social networks, $R(m, n)$ stands for the minimum number of people you would need to invite to your party to guarantee that either m of them are mutual acquaintances or n of them are mutual strangers.

8. What is $R(2, 3)$? (Can you solve the $(2, 3)$-Puzzle on two vertices? What about three vertices?)
9. What is $R(2, 4)$? (Can you solve the $(2, 4)$-Puzzle on three vertices? What about four vertices?)
10. What is $R(2, n)$ for any whole number n? (Can you solve the $(2, n)$-Puzzle on $n - 1$ vertices? What about n vertices?)

We can modify our "majority" argument from earlier to prove $R(3,4) \leq 12$. In words, we will prove the $(3,4)$-Puzzle on twelve vertices is unsolvable.

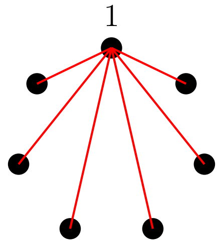
Restrict your attention for a moment to vertex 1. It has eleven edges connected to it. No matter how these edges are colored, there must be a blue majority or a red majority.

If there is a blue majority, then there are at least six blue edges connected to vertex 1.



11. Try adding the edges within the K_6. Will you always form a blue K_3 or a red K_4? Why?

If there is a red majority, then there are at least six red edges connected to vertex 1.



12. Try adding the edges within the K_6. Will you always form a blue K_3 or a red K_4? Why? (For this one, it is helpful to remember that $R(3,3) = 6$.)

In either case (blue majority or red majority), we cannot complete the $(3,4)$-Puzzle. Since we are working on a twelve vertex board, we have proven $R(3,4) \leq 12$.

13. The exact value for $R(3,4)$ is actually lower than 12. See if you can revise your proof to make it work with fewer vertices.

Please wait for the moderator before proceeding.

Every Ramsey Number Exists

We've proven some results about a handful of Ramsey numbers so far, but does $R(m, n)$ exist for any whole numbers m and n? For example, does the $(1000, 1000)$-Puzzle become unsolvable for a large enough number of vertices? In other words, can we really throw a large enough party that guarantees either 1,000 mutual acquaintances or 1,000 mutual strangers? (Caveat: The number of people we would need to invite far exceeds the number of people who have ever lived.)

We will make use of a technique called **recursion**, which means we use smaller Ramsey numbers to prove that larger Ramsey numbers exist. We've actually used this technique already - we used the fact that $R(3, 3)$ exists to prove that $R(3, 4)$ exists. Let's take one more step and prove that $R(4, 4)$ exists. To do so, we rely on the fact that $R(3, 4)$ and $R(4, 3)$ exist (in fact, they are equal to each other).

Proposition. $R(4, 4) \leq R(3, 4) + R(4, 3)$, *and therefore* $R(4, 4)$ *exists.*

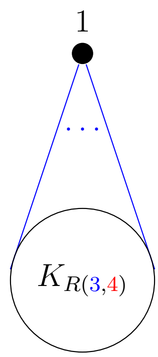
Take a moment to digest what the inequality says. Remember $R(3, 4)$ and $R(4, 3)$ are just numbers. (We proved that this number is no bigger than 12, and in fact the correct value is smaller.) The proposition says that the $(4, 4)$-Puzzle on $R(3, 4) + R(4, 3)$ vertices is unsolvable.

Let us trying solving the $(4, 4)$-Puzzle on $R(3, 4) + R(4, 3)$ vertices. Restrict your attention for a moment to vertex 1. It has $R(3, 4) + R(4, 3)$ - 1 edges connected to it. No matter how the graph is colored, *one* of the following two statements must be true:

- at least $R(3, 4)$ of these edges is blue, *or*
- at least $R(4, 3)$ of these edges is red.

This seems strange at first, but believe for a moment that both statements are false. According to this belief, there are at most $R(3, 4) - 1$ blue edges and at most $R(4, 3) - 1$ red edges, which is only $R(3, 4) + R(4, 3)$ - 2 edges in total. Since the belief leads to a contradiction ($R(3, 4) + R(4, 3)$ - 2 edges instead of the required $R(3, 4) + R(4, 3)$ - 1), we reject it.

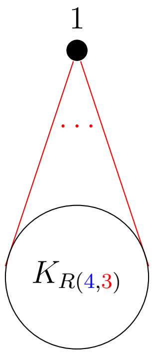
Consider the case in which there are at least $R(3, 4)$ blue edges connected to vertex 1.



14. According to the definition of $R(3, 4)$, one of two structures must appear in the lower circle. What are they?

15. Based on the previous answer, can you guarantee that there is either a blue K_4 or a red K_4 in the overall picture? (Don't forget about vertex 1.)

In the remaining case, there are at least $R(4,3)$ red edges connected to vertex 1.



16. According to the definition of $R(4,3)$, one of two structures must appear in the lower circle. What are they?
17. Based on the previous answer, can you guarantee that there is either a blue K_4 or a red K_4 in the overall picture? (Don't forget about vertex 1.)

If we are careful with our record-keeping, we'll notice that we've exhausted every possibility for colorations of this graph. In each case, we discover either a blue K_4 or a red K_4, which means $R(4,4) \leq R(3,4) + R(4,3)$.

We can duplicate this proof as many times as we like, each time stepping up the size of the blue or red complete graph we desire. For example, we might prove all the Ramsey numbers with a blue K_3 exist as follows:

- $R(3,5) \leq R(2,5) + R(3,4)$, so $R(3,5)$ exists.
- $R(3,6) \leq R(2,6) + R(3,5)$, so $R(3,6)$ exists.
- $R(3,7) \leq R(2,7) + R(3,6)$, so $R(3,7)$ exists.

Notice each subsequent proof uses a Ramsey number that has just been shown to exist.

18. Replace the question marks to make a reasonable conjecture: $R(4,5) \leq R(?,?) + R(?,?)$.
19. Mimic the proof we used for $R(4,4)$ to prove your conjecture about $R(4,5)$.

While these individual proofs are nice to help us understand Ramsey numbers, a mathematician would never dream of proving all these claims one by one. There are an infinite number of possibilities, and we don't have time to prove each one separately. The technique of **mathematical induction** allows one to prove an infinite number of statements at once. Using this technique, we could show that every Ramsey number exists with a single proof that is extremely similar to the one we gave for $R(4,4)$.

Theorem. *For any whole numbers m and n,*

$$R(m,n) \leq R(m-1,n) + R(m,n-1),$$

and therefore $R(m,n)$ exists.

Please wait for the moderator before proceeding.

More About Ramsey Numbers

- To prove a statement like $R(3,3) > 5$, we produced a specific coloration of K_5 that contained neither a blue K_3 nor a red K_3. As you can imagine, this becomes very difficult to accomplish for large Ramsey numbers. Paul Erdős (who is an interesting character in his own right) is perhaps most famous for his development of the **probabilistic method**, which allows one to prove that an object exists without actually constructing it. (If it sounds like magic, that's because it is.) In the context of Ramsey numbers, Erdős showed that there is a coloration of the complete graph on $2^{\frac{m-1}{2}}$ vertices that contains neither a blue K_m nor a red K_m, which means $R(m,m) > 2^{\frac{m-1}{2}}$ for every whole number m. His proof shows that such a coloration *exists*, but does not tell what it actually looks like, which is simultaneously the most beautiful and the most frustrating aspect of the probabilistic method. Learn more: tinyurl.com/ramseylower

- While we're on the subject of $R(m,m)$, the theorem at the end of the previous section can be used to provide an explicit (rather than recursive) upper bound. Recall that $R(m,n) = R(n,m)$, since we are just switching what we mean by "blue" and "red". According to the theorem,

$$R(m,m) \leq R(m-1,m) + R(m,m-1) = 2R(m-1,m).$$

  Essentially, this means it "costs" a factor of 2 to increase one of the colors by a single vertex, and thus a factor of 4 jump from $R(m-1,m-1)$ to $R(m,m)$. With a little care, one can use this observation to prove $R(m,m) \leq 4^{m-1}$. This means that every coloration of $K_{4^{m-1}}$ must contain a monochromatic K_m.

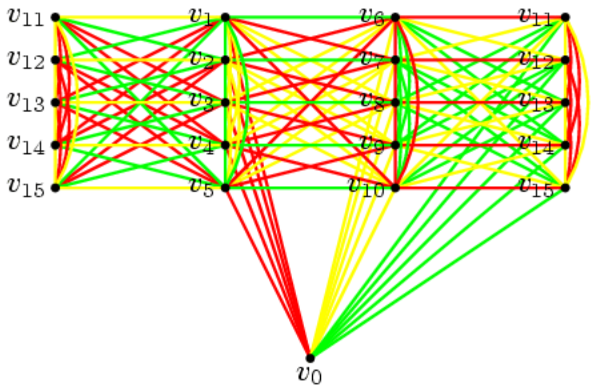
- Taking the first two points together, we know

$$2^{\frac{m-1}{2}} < R(m,m) \leq 4^{m-1}.$$

  Even though many talented mathematicians have been thinking about this problem for nearly 100 years, no substantial improvements have been made to these bounds.

- Very few Ramsey numbers are known exactly, but mathematicians work on narrowing the range of particular values. For example, it is known that $40 \leq R(3,10) \leq 42$. A straightforward computer search through all the colorations on even 40 vertices would require much longer than the age of the universe to complete, so it will take considerable ingenuity to determine $R(3,10)$ exactly. Learn more: tinyurl.com/smallramsey

- One could ask about Ramsey numbers that use more than two colors. For example, $R(3,4,5)$ would be the number of vertices needed so that every coloration of the complete graph contains either a yellow K_3, a red K_4, or a green K_5. The only three-color Ramsey number that is known exactly is $R(3,3,3) = 17$. The following coloration of K_{16} has no monochromatic triangle in any of the three colors and gives some sense of how complicated it is to work with more than two colors.



  Essentially the same proof that shows Ramsey numbers always exist for two colors can be used to show Ramsey numbers always exist for *any* number of colors. For example, every coloration of a big enough complete graph using 1,000 different colors must contain a monochromatic $K_{1,000,000}$. Learn more: tinyurl.com/ramsey333

- Allow me to write "K_∞" to mean "a complete graph on an infinite number of vertices". There is an infinite version of Ramsey's theorem with a marvelous proof: Every coloration of K_∞ contains a monochromatic K_∞. Order from chaos, indeed. Learn more: tinyurl.com/ramseyinfinite