Abstract The Lovász local lemma is a well-known probabilistic technique commonly used to prove the existence of rare combinatorial objects. We explore the lopsided (or negative dependency graph) version of the lemma, which, while more general, appears infrequently in literature due to the lack of settings in which the additional generality has thus far been needed. We present a general framework (matchings in hypergraphs) from which many such settings arise naturally. We also prove a seemingly new generalization of Cayley’s formula, which helps defining negative dependency graphs for extensions of forests into spanning trees. We formulate open problems regarding partitions and doubly stochastic matrices that are likely amenable to the use of the lopsided local lemma.

1 Introduction

The Lovász Local Lemma (hereinafter, “L3”) says there is a positive probability that none of the events in some collection occurs, provided the dependencies among them are not too strong. First we state L3 [3, 7, 22, 23]. Given events $A_1, \ldots, A_n$ in a probability space, define a dependency graph to be a simple graph $G$ on $V(G) = [n]$ (i.e., the set of the names of the events, $\{1, 2, \ldots, n\}$) satisfying that every event $A_i$ is independent of the elements of the event algebra generated by $\{A_j : ij \notin E(G)\}$. 
Lemma 1 (Lovász Local Lemma). [3] Let $A_1, \ldots, A_n$ be events with dependency graph $G$. If there are numbers $x_1, \ldots, x_n \in [0, 1)$ such that for all $i$

$$\Pr(A_i) \leq x_i \prod_{ij \in E(G)} (1 - x_j).$$

Then

$$\Pr\left(\bigwedge_{i=1}^n \overline{A_i}\right) \geq \prod_{i=1}^n (1 - x_i) > 0.$$ 

The original lemma of Lovász in [7], which is a special case of Lemma 1, was used to prove the 2-colorability of a certain hypergraph, and in general, together with the extension L3 [22], to prove the existence of combinatorial objects, for which construction seemed hopeless. Surprisingly, there are recent constructive versions of the lemma (e.g., in [19]) that will produce the desired object explicitly for many coloration problems. Define $C_G(i) = \{j \in [n] \mid ij \notin E(G)\}$. Inspection of the—not too difficult—proof of L3 concludes that we use independence in the form of

$$\forall i \in [n] \forall S \subseteq C_G(i) \quad \Pr\left(\bigwedge_{j \in S} A_j\right) \neq 0 \rightarrow \Pr\left(\bigwedge_{j \in S} A_j\bigg| A_i\right) = \Pr(A_i). \quad (1)$$

Further analyzing the proof, it turns out that the inequality below is sufficient, instead of equality in (1):

$$\forall i \in [n] \forall S \subseteq C_G(i) \quad \Pr\left(\bigwedge_{j \in S} A_j\right) \neq 0 \rightarrow \Pr\left(A_i\bigg| \bigwedge_{j \in S} \overline{A_j}\right) \leq \Pr(A_i). \quad (2)$$

Formally, a negative dependency graph for given events $A_1, \ldots, A_n$ in a probability space is a simple graph $G$ on $V(G) = [n]$ (i.e., the set of the names of the events, $\{1, 2, \ldots, n\}$) satisfying (2). Observe that every negative dependency graph is a dependency graph, but not vice versa.

Lemma 2 (Lopsided Lovász Local Lemma, L4). Lemma 1 holds with “dependency graph” relaxed to “negative dependency graph.”

The inequality in (2) explains the term “lopsided” in the literature for L4. Negative dependency graphs and the lopsided version of the lemma were first introduced in [8] by Erdős and Spencer to prove the existence of a certain latin transversal, and independently by Albert, Frieze, and Reed [2], following the original Lovász version [7]. L4 as stated above is due to Ku [14]. These papers did not investigate classes of problems where L4 could be applied. Few results are present in the literature making use of a negative dependency graph that is not also a dependency graph.

It is worth mentioning here the following useful equivalents of (2):

$$\forall i \in [n] \forall S \subseteq C_G(i) \quad \Pr(A_i) \neq 0 \rightarrow \Pr\left(\bigwedge_{j \in S} \overline{A_j}\bigg| A_i\right) \leq \Pr\left(\bigwedge_{j \in S} \overline{A_j}\right), \quad (3)$$
or

\[ \forall i \in [n] \forall S \subseteq C_G(i) \quad \Pr \left( \bigwedge_{j \in S} \overline{A_j} \right) \neq 0 \rightarrow \Pr (\overline{A_i}) \leq \Pr \left( \bigwedge_{j \in S \cup \{i\}} \overline{A_j} \right), \quad (4) \]

and also the following, which takes the form of a correlation inequality:

\[ \forall i \in [n] \forall S \subseteq C_G(i) \quad \Pr (A_i) \Pr (\bigvee_{j \in S} A_j) \leq \Pr (A_i \land (\bigvee_{j \in S} A_j)). \qquad (5) \]

We note that even more general versions of the lemma hold practically with the same proof. Given events \(A_1, \ldots, A_n\) in a probability space, define a dependency digraph \(\overrightarrow{G}\) on \(V(G) = [n]\) by requiring that every event \(A_i\) is independent of the elements of the event algebra generated by \(\{A_j : j \in C_{\overrightarrow{G}}(i)\}\), with \(C_{\overrightarrow{G}}(i) = \{j : ij \notin E(\overrightarrow{G})\}\), and a negative dependency digraph graph \(\overrightarrow{G}\) by (2), using \(\overrightarrow{G}\) and \(C_{\overrightarrow{G}}(i)\) instead of \(G\) and \(C_G(i)\) in (2).

**Lemma 3 (Lopsided Lovász Local Lemma, digraph version \(\overrightarrow{L4}\) [3]).** Lemma 1 holds with “dependency graph” relaxed to “negative dependency digraph.”

## 2 Examples of Negative Dependency Graphs

### 2.1 Random Matchings in Complete Uniform Hypergraphs

A matching \(M\) in a hypergraph is a collection of pairwise vertex-disjoint hyperedges. The set of vertices covered by hyperedges of the matching \(M\) is denoted by \(V(M)\). A matching is *perfect* if every vertex of the underlying hypergraph appears in some hyperedge of the matching. In what follows, we will restrict our attention to matchings of the complete \(k\)-uniform hypergraph on \(N\) vertices, commonly denoted \(K^k_N\).

A matching of \(K^k_N\) is an \(r\)-matching if it consists of \(r\) vertex-disjoint hyperedges. For some fixed integer \(k, r, N\) satisfying \(k \geq 2, r \geq 1,\) and \(N \geq rk\), let \(\Omega^k_{N,r}\) be the uniform probability space over all \(r\)-matchings of \(K^k_N\). An \(r\)-matching of \(K^k_N\) is maximal if \(r = \lceil \frac{N}{k} \rceil\); it is perfect if \(N = kr\). The space of maximal matchings of \(K^k_N\) is simply denoted by \(\Omega^k_N\).

Given a matching \(M\) in \(K^k_N\), we will be interested in the canonical event \(A^M\) containing all \(r\)-matchings that extend \(M\). More precisely,

\[ A^M = A^M_{N,k,r} = \left\{ M' \in \Omega^k_{N,r} \mid M \subseteq M' \right\}. \]
We will say that two matchings conflict, if they have edges that are neither identical nor disjoint, two canonical events conflict, whenever the matchings used to define them conflict.

Given a collection \( \mathcal{M} \) of matchings (or other combinatorial objects, as we will see later), the conflict graph for the collection \( \{ A^M \mid M \in \mathcal{M} \} \) is the simple graph with vertex set \( \mathcal{M} \) and edge set \( \{ M_1M_2 \mid M_1 \in \mathcal{M} \text{ and } M_2 \in \mathcal{M} \text{ are in conflict} \} \). The following theorem was proved in [17] in the special case \( k = 2 \) and \( N \) is even:

**Theorem 1.** Let \( \mathcal{M} \) be a collection of matchings in \( K^k_N \). The conflict graph for the collection of canonical events \( \{ A^M \mid M \in \mathcal{M} \} \) is a negative dependency graph for the probability space \( \Omega_N^k \).

Theorem 1 can be deduced from the following lemma, which deals with a more general probability space \( \Omega_N^{k,r} \). The proof of the lemma is postponed.

**Lemma 4.** In \( \Omega_N^{k,r} \), fix a matching \( M \in \mathcal{M} \) with \( |M| < r \). Let \( \mathcal{J} \) be any collection of matchings from \( \mathcal{M} \) whose members do not conflict with \( M \). If \( |\bigcup_{M' \in \mathcal{J}} V(M' \setminus M)| \leq rk - |V(M)| + k - 1 \) and \( \Pr(\bigwedge_{M' \in \mathcal{J}} \overline{A^{M'}}) > 0 \), then

\[
\Pr\left( A^M \left| \bigwedge_{M' \notin \mathcal{J}} \overline{A^{M'}} \right. \right) \leq \Pr(A_M).
\]

**Proof of Theorem 1:** Write \( N = rk + t \). For the space \( \Omega_N^k \) of maximum matchings, we have \( 0 \leq t \leq k - 1 \). In this case, we have

\[
|\bigcup_{M' \in \mathcal{J}} V(M' - M)| \leq N - |V(M)| = rk + t - |V(M)| \leq rk - |V(M)| + k - 1.
\]

By Lemma 4, the conflict graph for the collection of canonical events \( \{ A^M \mid M \in \mathcal{M} \} \) is a negative dependency graph for \( \Omega_N^k \). □

The following example shows the conflict graph may not be a negative dependency graph of \( \Omega_N^{k,r} \) for \( r < \lfloor \frac{N}{k} \rfloor \). Take \( M_1, M_2 \in \Omega_N^{k,r} \) so that \( M_1 \cup M_2 \) is a matching of size \( r + 1 \). This is possible since \( r + 1 \leq \lfloor \frac{N}{k} \rfloor \). By our choice, \( M_1 \) and \( M_2 \) are not adjacent in the conflict graph. However, we have

\[
\Pr(A_{M_1}|A_{M_2}) = 1 > \Pr(\overline{A_{M_1}}),
\]

contradicting (3). Nevertheless, we can add orientation and some edges to the conflict graph to turn it into a negative dependency digraph.

**Theorem 2.** For integers \( r \geq 1, k \geq 2, \) and \( N \geq (r + 1)k \), let \( \mathcal{M} \) be a collection of \( r \)-matchings in \( K^k_N \). For any \( M \in \mathcal{M} \), let \( S_M \) be an arbitrary set of \( N - (r + 1)k + 1 \) vertices not in \( V(M) \). Let \( G \) be a graph on the vertex set \( \mathcal{M} \) where \( MM' \) is a directed edge of \( G \) if \( M' \) conflicts with \( M \) or \( M' \) contains some vertex in \( S_M \). Then \( G \) is a negative dependency digraph for the collection of canonical events \( \{ A^M \mid M \in \mathcal{M} \} \) in the probability space \( \Omega_N^{k,r} \).
Throughout, we assume that the vertex set of $K^k_N$ is $[N]$. We will view the identity map as an injection from $[N]$ into $[N + s]$ for $s \geq 0$ and also from $V(K^k_N)$ to $V(K^k_{N+s})$ and from $E(K^k_N)$ to $E(K^k_{N+s})$. To emphasize the difference in the probability space, we use $A^M_{N,k,r}$ to denote the canonical event induced by the matching $M$ in $\Omega^r_N$ and use $\Pr^r_N(\bullet)$ to denote the probability in $\Omega^r_N$. To simplify our notation, we will write $\Pr(\bullet)$ for $\Pr^r_N(\bullet)$, if the probability space is $\Omega^r_N$, and spell out the full notation otherwise.

Lemma 5. For any integers $r \geq 1$, $k \geq 2$, $N \geq rk$, $N_0 = \min\{N - k, rk - 1\}$ and any collection $\mathcal{M}$ of matchings in $K^k_N$, we have

$$
\Pr^r_{N-k}\left(\bigwedge_{M \in \mathcal{M}} A^M_{N-k,k,r-1}\right) \leq \Pr^r_{N}\left(\bigwedge_{M \in \mathcal{M}} A^M_{N,k,r}\right). \tag{7}
$$

Proof. Let $t = N - rk$. We have $0 \leq t \leq N - k$. Given an $r$-matching $M'' \in \Omega^r_N$, $M''$ can be viewed as a hypergraph with $r$ pairwise disjoint hyperedges and $t$ isolated vertices. The right end of a hyperedge $H$ is max$_{i \in H} i$. A hyperedge of $M''$ is called rightmost if it has the largest right end among all hyperedges in $M''$. A $k$-edge $R$ can be the rightmost hyperedge of some matching $M'' \in \Omega^r_N$ if and only if the right end vertex of $R$ is at least $rk$. For $i = 0, 1, 2, \ldots, t$, let $\mathcal{R}_i$ be the family of $k$-edges whose right end is $N - i$. Let $\mathcal{R} = \cup_{i=0}^t \mathcal{R}_i$. Consider the mapping $\psi: \Omega^r_N \to \mathcal{R}$, which maps $M''$ to its rightmost hyperedge. Clearly, $\cup_{R \in \mathcal{R}} \psi^{-1}(R)$ forms a partition of $\Omega^r_N$.

Fix an $i$ ($0 \leq i \leq t$) and a $k$-edge $R \in \mathcal{R}_i$. Easy calculation shows that

$$
a_i := \Pr(\psi^{-1}(R)) = \frac{k!r(t)_i}{(N)_i(N-i)_k}.
$$

Direct comparison of terms gives

$$
a_0 \geq a_1 \geq \cdots \geq a_t. \tag{8}
$$

Since $N_0 \leq kr - 1$, the hyperedge $R$ above is not in any matching $M' \in \mathcal{M}$. Define $\mathcal{M}'(R) = \{M' \in \mathcal{M}: V(M') \cap R = \emptyset\}$ and observe that

$$
\bigwedge_{M' \in \mathcal{M}} A^M_{N,k,r} \land \psi^{-1}(R) = \bigwedge_{M' \in \mathcal{M}'(R)} A^M_{N,k,r} \land \psi^{-1}(R).
$$

Let $F = \{N - k + 1, N - k + 2, \ldots, N\}$ and $\sigma$ be any permutation of $[N]$ that maps $R \backslash F$ to $F \backslash R$, maps $F \backslash R$ to $R \backslash F$, and leaves other vertices as fixed points.
The permutation $\sigma$ maps $\bigwedge_{M' \in \mathcal{M}} \overline{A_{N,k,r}^{M'}} \land \psi^{-1}(R)$ to $\bigwedge_{M' \in \mathcal{M}} \overline{A_{N,k,r}^{M'}} \land A_{N,k,r}^F$.

We have

$$
\Pr \left( \bigwedge_{M' \in \mathcal{M}} \overline{A_{N,k,r}^{M'}} \right) = \sum_{i=0}^{t} \sum_{R \in \mathcal{R}_i} \Pr \left( \bigwedge_{M' \in \mathcal{M}} \overline{A_{N,k,r}^{M'}} \land \psi^{-1}(R) \right)
$$

$$
= \sum_{i=0}^{t} \sum_{R \in \mathcal{R}_i} \Pr \left( \bigwedge_{M' \in \mathcal{M}} \overline{A_{N,k,r}^{M'}} \land A_{N,k,r}^F \right)
$$

$$
\geq \sum_{i=0}^{t} \sum_{R \in \mathcal{R}_i} \Pr \left( \bigwedge_{M' \in \mathcal{M}} \overline{A_{N,k,r}^{M'}} \land A_{N,k,r}^F \right) \Pr \left( A_{N,k,r}^F \right)
$$

$$
= \Pr_{N-k}^{k,r-1} \left( \bigwedge_{M' \in \mathcal{M}} \overline{A_{N-k,k,r-1}^{M'}} \right) \sum_{i=0}^{t} \sum_{R \in \mathcal{R}_i} \Pr \left( A_{N,k,r}^F \right).
$$

In the last step, we use the fact that

$$
\Pr_{N}^{k,r} \left( \bigwedge_{M' \in \mathcal{M}} \overline{A_{N,k,r}^{M'}} \right) = \Pr_{N-k}^{k,r-1} \left( \bigwedge_{M' \in \mathcal{M}} \overline{A_{N-k,k,r-1}^{M'}} \right).
$$

Note that $\Pr(A_{N,k,r}^F) = a_0 \geq a_i = \Pr \left( \psi^{-1}(R) \right)$. We have

$$
\sum_{i=0}^{t} \sum_{R \in \mathcal{R}_i} \Pr \left( A_{N,k,r}^F \right) \geq \sum_{i=0}^{t} \sum_{R \in \mathcal{R}_i} \Pr \left( \psi^{-1}(R) \right) = \Pr \left( \Omega_N^{k,r} \right) = 1.
$$

Thus,

$$
\Pr \left( \bigwedge_{M' \in \mathcal{M}} \overline{A_{N,k,r}^{M'}} \right) \geq \Pr_{N-k}^{k,r-1} \left( \bigwedge_{M' \in \mathcal{M}} \overline{A_{N-k,k,r-1}^{M'}} \right).
$$

\[ \square \]

**Proof of Lemma 4:** Returning to the proof of Lemma 4, Fix a matching $M \in \mathcal{M}$ and let $\mathcal{J}$ be any collection of matchings from $\mathcal{M}$ that do not conflict with $M$. Our aim is to show

$$
\Pr \left( A^M \bigg| \bigwedge_{M' \in \mathcal{J}} \overline{A_{N,k,r}^{M'}} \right) \leq \Pr \left( A^M \right).
$$
Observe that the inequality holds trivially when \( \Pr(A^M) = 0 \). Otherwise, the above formula is equivalent with the following (that is essentially (3)):

\[
\Pr\left( \bigwedge_{M' \in \mathcal{J}} \overline{A^{M'}} \bigg| A^M \right) \leq \Pr\left( \bigwedge_{M' \in \mathcal{J}} \overline{A^{M'}} \right).
\]

Let \( \mathcal{J}^M = \{ M' \setminus M \mid M' \in \mathcal{J} \} \). If \( M \in \mathcal{J} \), then the left-hand side of the estimate above is zero, and so we have nothing to do. Assume instead that \( M \notin \mathcal{J} \). Since every matching \( M' \) in \( \mathcal{J} \) is not in conflict with \( M \), the vertex set of \( M' \setminus M \) is nonempty and is disjoint from the vertex set of \( M \). Let \( T \) be the set of vertices covered by the matching \( M \) and \( U \) be the set of vertices covered by at least one matching \( F \in \mathcal{J}^M \). We have \( T \cap U = \emptyset \). Let \( \pi \) be a permutation of \([N]\) mapping \( T \) to \([N - |T| + 1, N - |T| + 2, \ldots, N]\). We have \( \pi(T) \cap \pi(U) = \emptyset \). Thus, \( \pi(U) \subseteq [N - |T|] \). Define \( \pi(\mathcal{J}^M) \) to be the collection \( \{ \pi(F) \mid F \in \mathcal{J}^M \} \). We obtain

\[
\Pr\left( \bigwedge_{M' \in \mathcal{J}} \overline{A^{M'}} \bigg| A^M \right) = \frac{\Pr\left( \bigwedge_{M' \in \mathcal{J}} A^{M'} \cap A^M \right)}{\Pr(A^M)} = \frac{\Pr\left( \bigwedge_{M' \in \mathcal{J}} A^{M'} \cap A^M \right)}{\Pr(A^M)} = \frac{\Pr\left( \bigwedge_{F \in \mathcal{J}^M} A^F \cap A^M \right)}{\Pr(A^M)} = \frac{\Pr\left( \bigwedge_{F \in \mathcal{J}^M} A^F \bigg| A^M \right)}{\Pr(A^M)} \leq \Pr\left( \bigwedge_{F \in \mathcal{J}^M} A^F \bigg| A^M \right) \text{ (by Lemma 5)}
\]

\[
= \Pr\left( \bigwedge_{F \in \mathcal{J}^M} A^F \bigg| A^M \right)
\]
\[= \Pr \left( \bigwedge_{M' \in \mathcal{J}} A_{M', k, r}^M \right) \]
\[\leq \Pr \left( \bigwedge_{M' \in \mathcal{J}} A_{M', k, r}^M \right). \]

2.2 Random Matchings in Complete Multipartite Graphs

Theorem 1 shows how a general class of negative dependency graphs can arise from the space of random perfect matchings of \(K_N^k\). A similar result was shown in [16] for the uniform probability space of maximum matchings of a complete bipartite graph \(K_{s,t}\), with the same definition of “conflict” and “canonical event” as above. This can be viewed as the uniform probability space of random injections from an \(s\)-element set into a \(t\)-element set (for \(s \leq t\)), providing a plethora of applications.

This generalizes to multipartite matchings as follows. For details see [18]. Let us be given disjoint sets \(U_1, \ldots, U_m\) with \(|U_1| \leq |U_i|\) for \(1 < i\). Call edges the sets \(H\), if \(H \subseteq \bigcup_{i=1}^m U_i\) and for all \(i\), \(|H \cap U_i| = 1\). A matching is a set of disjoint edges. A matching is of maximum size if it covers all elements of \(U_1\). Consider the uniform probability measure on set of all maximum size matchings. Given a matching \(M\), let \(A^M\) denote the event of all maximum size matchings that contain all edges of \(M\). We say that two matchings, \(M_1\) and \(M_2\), are in conflict if they contain edges that are neither identical nor disjoint.

\[\text{Theorem 3. [18] Let } \mathcal{M} \text{ be a collection of multipartite matchings on } U_1, \ldots, U_m. \text{ The conflict graph for the collection of canonical events } \{A^M | M \in \mathcal{M}\} \text{ is a negative dependency graph.}\]

2.3 Spanning Trees in Complete Graphs

The various matching spaces we have mentioned have in common that a partial matching does not conflict with any element of its corresponding canonical event. Indeed, the proof of Theorem 1 relies heavily on this fact by reducing the problem of extending a given partial matching to the problem of finding matchings on the unmatched vertices only.

Consider now the uniform probability space of all spanning trees of \(K_N\). Given a forest \(F\) (i.e., a cycle-free subset of the edges of \(K_N\)), the canonical event \(A^F\) is the collection of all spanning trees of \(K_N\) containing \(F\). We say that two forests conflict whenever there are a pair of edges, one in the first forest and one in the second, that intersect in exactly one vertex. In other words, two forests \(F\) and \(F'\) do not conflict if for every connected component \(C \subseteq F\) and \(C' \subseteq F'\), \(C\) and \(C'\) are either identical or disjoint.
Theorem 4. Let \( \mathcal{F} \) be a collection of forests in \( K_N \). The conflict graph for the collection of canonical events \( \{ A^F \mid F \in \mathcal{F} \} \) is a negative dependency graph.

Notice that the spanning tree setting stands in stark contrast to the matching and partition settings; a forest conflicts with every spanning tree in its corresponding canonical event! The proof of Theorem 4 hinges on two lemmata. The first is a direct generalization of Cayley’s theorem, while the second is a special case of Theorem 4.

Lemma 6. Let us be given a forest \( F \) in \( K_N \), which has its components \( C_1, C_2, \ldots, C_m \) on \( f_1, f_2, \ldots, f_m \) vertices. Then, the number of spanning trees \( T \) in \( K_N \), such that \( F \) is contained by \( T \), is

\[
f_1 f_2 \cdots f_m N^{N-2-\sum (f_i-1)}.
\]

(9)

Proof. Recall Menon’s Theorem (Problem 4.1 in [15]): the number of spanning trees in \( K_N \) with prescribed degrees \( d_1, d_2, \ldots, d_N \) in vertices \( 1, 2, \ldots, N \) is the multinomial coefficient \( \binom{N-2}{(d_1-1), \ldots, (d_N-1)} \). Contracting the components of \( F \) to single vertices, \( T \) contracts to a spanning tree \( T^* \) of \( K_N - \Sigma_i (f_i-1) \). Let \( v_1, \ldots, v_m \) denote the result of contraction of \( C_1, C_2, \ldots, C_m \), and \( u_1, u_2, \ldots, u_{N-\Sigma_i f_i} \) the vertices from \( 1, 2, \ldots, N \), not covered by \( F \). By Menon’s theorem, the number of \( H \) spanning trees of \( K_N - \Sigma_i (f_i-1) \), with degree \( d_i \) in \( v_i \) and \( D_j \) in \( u_j \), is

\[
\binom{N-2-\sum (f_i-1)}{(d_1-1), \ldots, (d_m-1), (D_1-1), \ldots, (D_{N-\Sigma_i f_i}-1)}.
\]

Note that every \( H \) spanning tree of \( K_N - \Sigma_i (f_i-1) \) with degree \( d_i \) in \( v_i \) and \( D_j \) in \( u_j \) arises precisely \( \prod_i f_i^{d_i} \) ways as a contraction \( T^* \) from some \( T \) spanning tree of \( K_N \). Hence the number of spanning trees \( T \) containing \( F \) is

\[
\sum \binom{N-2-\sum (f_i-1)}{(d_1-1), \ldots, (d_m-1), (D_1-1), \ldots, (D_{N-\Sigma_i f_i}-1)} \prod_i f_i^{d_i},
\]

where the summation goes for all \( d_1, \ldots, d_m \) and \( D_1, \ldots, D_{N-\Sigma_i f_i} \) sequences. The multinomial theorem easily evaluates this summation to the required quantity. □

For the next lemma, we say two forests are in **strong conflict**, if they are not vertex disjoint.

Lemma 7. Let \( \mathcal{F} \) be a collection of forests in \( K_N \). The strong conflict graph for the collection of canonical events \( \{ A^F \mid F \in \mathcal{F} \} \) is a negative dependency graph.

Proof. To prove Lemma 7, we prove (5), where \( A_i \) is the set of spanning trees containing the forest \( F_i \), and \( F_i \) is not in strong conflict with \( F_j \) for any \( j \in S \).

By inclusion–exclusion,

\[
\Pr \left( \bigvee_{j \in S} A_j \right) = \sum_{R \subseteq S, |R| \geq 1} \Pr \left( \bigwedge_{j \in R} A_j \right) (-1)^{|R| - 1}
\]
and

\[
\Pr\left( A_i \land \left( \bigvee_{j \in S} A_j \right) \right) = \sum_{R \subseteq S, |R| \geq 1} \Pr\left( A_i \land \left( \bigwedge_{j \in R} A_j \right) \right) (-1)^{|R|-1}.
\]

Observe that the event \( A_i \land \left( \bigwedge_{j \in R} A_j \right) \) consists of spanning trees that contain the forest \( F_i \) and \( G_R = \bigcup_{j \in R} F_j \). The latter graph is either a forest or contains cycle. In the latter case, the corresponding event is impossible. Finally, we claim

\[
\Pr(A_i) \Pr\left( \bigwedge_{j \in R} A_j \right) = \Pr\left( A_i \land \left( \bigwedge_{j \in R} A_j \right) \right)
\]

either by \( G_R \) being impossible (and both sides are zero) or by \( F_i \) and \( G_R \) being vertex-disjoint forests, whose union is a forest again, having as components each and every component of \( F_i \) and \( G_R \). Lemma 6 finishes the proof.

Finally, to prove Theorem 4, let \( A_i \) denote again the set of spanning trees containing the forest \( F_i \). We are going to prove (5). If \( F_i \) has no strong conflict with any \( F_j \) \((j \in S)\), then Lemma 7 already gives us the wanted result. Now suppose \( F_i \) does have strong conflict with some \( F_j \) \((j \in S)\). Define \( F_j' = F_j \setminus F_i \) (we mean the difference of the edge sets). Let \( A_j' \) be the event corresponding to \( F_j' \). We have

\[
\Pr\left( A_i \land \left( \bigwedge_{j \in S} \overline{A}_j \right) \right) = \Pr\left( A_i \land \left( \bigwedge_{j \in S} \overline{A}_j' \right) \right)
\]

\[
= \Pr(A_i) \Pr\left( \bigwedge_{j \in S} \overline{A}_j' \right) \quad \text{(by Lemma 7)}
\]

\[
\leq \Pr(A_i) \Pr\left( \bigwedge_{j \in S} \overline{A}_j \right).
\]

Note that in Lemma 7 we proved a negative dependency graph with equalities everywhere. This is in fact a negative dependency graph. It is likely that independence is lurking in the form of independent choice of entries in the Prüfer code or some other sequence encoding of trees.

There is one more interesting comment to make here. Fix any connected graph \( G \) and two of its edges \( e \) and \( f \). In the uniform probability space of the spanning trees of \( G \), the correlation inequality

\[
\Pr(A^c) \Pr(A'^c) \geq \Pr(A^c \land A'^c)
\]

(10) holds \([25]\). This is the opposite of the inequality that we expect for (5)! There is no contradiction, however, as for \( G = K_N \) and disjoint edges, (10) holds with identity, and for two edges sharing a single vertex, we have a conflict and we made no claim.
Change the underlying probability space of spanning trees to the uniform probability space of spanning forests of $K_N$, and let the canonical event associated with a forest be the set of all spanning forests containing it. Then neither conflict nor strong conflict of forests defines a negative dependency graph for their canonical events—this is in line of the conjecture of Kahn [13] that in every connected graph, (10) holds, where $A^e$ is the set of spanning spanning forests containing edge $e$.

### 2.4 Upper Ideals in Distributive Lattices

Let $X$ be an $N$-element set, and let $\Omega_N$ be the probability space consisting of all subsets of $X$ and equipped with the uniform probability measure. For a fixed subset $Y$ of $X$, define the canonical event $A^Y$ to be the collection of all subsets of $X$ that contain $Y$. In other words,

$$A^Y = \{Z \in \Omega_N \mid Y \subseteq Z\}.$$

**Theorem 5.** Let $\mathcal{Y}$ be a collection of nonempty subsets of an $N$-element set. The graph with vertex set $\{A^Y \mid Y \in \mathcal{Y}\}$ is an edgeless negative dependency graph for the events $A^Y$.

More generally, let $\Gamma$ be a distributive lattice equipped with the uniform probability measure. For $Y \in \Gamma$, let

$$A^Y = \{Z \in \Gamma \mid Y \leq Z\}.$$

**Theorem 6.** Let $\mathcal{Y}$ be a collection of elements of a distributive lattice $\Gamma$. The graph with vertex set $\{A^Y \mid Y \in \mathcal{Y}\}$ is an edgeless negative dependency graph for the events $A^Y$.

**Proof.** Clearly Theorem 6 implies Theorem 5, if applied to the subset lattice. We have to show (5) for every $A_i = A^Y_i$ ($Y \in \mathcal{Y}$) and every $S \subseteq \Gamma \\setminus \{Y\}$. Consider the indicator functions of the sets $A^Y_i$ and $\vee_{U \in S}A^U$. These are increasing $\Gamma \rightarrow \mathbb{R}$ functions, to which the FKG inequality [10] applies, providing (5). Note that the FKG inequality follows from the even more general four functions theorem [1, 3]. The special case of (5) for the subset lattice already follows from [21].

### 2.5 Symmetric Events

We say that the events $A_1, A_2, \ldots, A_n$ are symmetric if the probability of any boolean expression of these events does not change if we substitute $A_{\pi(i)}$ to the place of $A_i$ simultaneously for any permutation $\pi$ of $[n]$. The following theorem was proved in [16]:
Theorem 7. Assume that the events $A_1, A_2, \ldots, A_n$ are symmetric, and let $p_i$ denote $\Pr(A_1 \land A_2 \land \ldots \land A_i)$ for $i = 1, 2, \ldots, n$, and let $p_0 = 1$. If the sequence $p_i$ is logconvex, i.e., $p_k^2 \leq p_{k-1}p_{k+1}$ for $k = 1, 2, \ldots, n - 1$, then these events have an empty negative dependency graph.

3 Open Problems

3.1 Maximum Size Matchings in Graphs

The concept of canonical event and conflict, as defined in Sect. 2.1, can be extended in the case $k = 2$ for maximum size matchings in any graph $G$. Theorems 1 (for $k = 2$) and 3 can be interpreted that for the graphs $G = K_n$ and $K_{s,t}$, conflict of canonical events defines a negative dependency graph. Not every ambient graph will allow this result [17]. For example, for $G = C_6$, let $e$ and $f$ be any two opposite edges. Notice there are only two perfect matchings in $C_6$. We have that $\Pr(A^{\{e\}}) = \frac{1}{2}$, while $1 = \Pr\left(A^{\{e\}} \mid A^{\{f\}}\right) \not\leq \Pr(A^{\{e\}})$. The 3-dimensional hypercube also fails this property. However, paths with even number of vertices have this property. Can we possibly classify the graphs that have this property?

3.2 Partition Lattice

The space of perfect matchings of $K_N^k$ can be viewed as the space of partitions of an $N$-element set in which every block is of size $k$. Can we still find a negative dependency graph without this restriction on block sizes? To state this question more precisely, we will call a collection of disjoint subsets of an $N$-element set a partial partition and say that two partial partitions conflict whenever they have two classes neither disjoint nor identical, i.e., their union is not again a partial partition. (A partial partition may in fact fully partition the underlying set.) The ambient probability space is the space of all partitions of an $N$-element set (equipped with the uniform distribution) so that the canonical event $A^M$ for a given partial partition $M$ is the collection of all partitions extending $M$.

Conjecture 1. Let $\mathcal{M}$ be a collection of partial partitions of an $N$-element. The conflict graph for the collection of canonical events $\{A^M \mid M \in \mathcal{M}\}$ is a negative dependency graph.

Despite its apparent similarity to Theorem 1, the proof we gave cannot be applied when there are no restrictions on the block sizes. In particular, the necessary adaptation of Lemma 5, namely

$$\Pr_N\left(\bigwedge_{M \in \mathcal{M}} \overline{A_N^M}\right) \leq \Pr_{N+1}\left(\bigwedge_{M \in \mathcal{M}} \overline{A_{N+1}^M}\right),$$
may fail in some instances. For example, let $M_1 = \{\{1\}, \{2\}\}$, $M_2 = \{\{1\}, \{3\}\}$, and $M_3 = \{\{2\}, \{3\}\}$. One can compute by hand that $\Pr(AM_1^M \land AM_2^M \land AM_3^M) = \frac{4}{5}$, while $\Pr(AM_4^M \land AM_2^M \land AM_3^M) = \frac{11}{15}$. Theorem 6 is not going to help as the partition lattice is not distributive.

Let $M, M_1, \ldots, M_k$ be partial partitions of an $N$-element set such that $M$ conflicts with none of the $M_i$ (but $M_i$ may conflict with $M_j$ for $i \neq j$). The required $\Pr(A^M \land \bigcap_{i=1}^{k} AM_i) \leq \Pr(A^M)$ is equivalent to the inequality $\Pr(A^M)\Pr(\bigvee_{i=1}^{k} AM_i) \leq \Pr(A^M \land (\bigvee_{i=1}^{k} AM_i))$ (see (5)).

Let $B_j$ denote the $j$th Bell number, which counts the number of partitions of a $j$-element set. The last inequality can be rewritten as

$$|A^M| \left| \bigcup_{i=1}^{k} AM_i \right| \leq B_N \left| A^M \cap \left( \bigcup_{i=1}^{k} AM_i \right) \right|.$$  

(11)

$|A_M| = B_{N-|\cup M|}$, and the other two terms in (11) can be expressed by Bell numbers using inclusion–exclusion; however, the expression will depend heavily on which blocks the partial partitions have in common and whether they conflict. If we assume that all the partial partitions $M_i$ have disjoint underlying sets and each cover exactly $m$ elements, (11) turns into

$$B_{N-m} \sum_{i=1}^{k} (-1)^{i+1} \binom{k}{i} B_{N-i \cdot m} \leq B_N \sum_{i=1}^{k} (-1)^{i+1} \binom{k}{i} B_{N-(i+1) \cdot m}.$$  

(12)

The inequality above has been verified asymptotically in $N$ for small fixed values of $m$ and $k$ with Maple, using the modification of the Moser–Wyman formula for the Bell numbers found in [5]. [5] says that uniformly for $h = O(\ln n)$, as $n \to \infty$,

$$B_n+h = \frac{(n+h)!}{(2n+2h)!} e^{h-1} \left( 1 + \frac{P_0 + hP_1 + h^2P_2}{e^h} + \frac{Q_0 + hQ_1 + h^2Q_2 + h^3Q_3 + h^4Q_4}{e^{2h}} + O(e^{-3h}) \right),$$

where $r = r^e = n$, $B = (r^2 + r)e^r$, $P_i$ and $Q_i$ are known rational functions of $r$. $P_i$ and $Q_i$ can be found explicitly in [4].

### 3.3 Permanent of Doubly Stochastic Matrices

Let $A = (a_{i,j})$ be an $n \times n$ doubly stochastic matrix with non-negative entries. For each $1 \leq i \leq n$, let $X_i$ be independent random variables that select the element $j$ from $\{1, \ldots, n\}$ with probability $a_{i,j}$. Define also $B_i$ to be the event that $X_i = X_j$ for some $j \neq i$.

**Conjecture 2.** The collection of events $\{B_i \mid 1 \leq i \leq n\}$ are the vertices of an edgeless negative dependency graph.

This conjecture is relevant because of the continuing interest in lower bounds for the permanent. Computing the permanent is #P-hard [26] and is hard for the entire
polynomial-time hierarchy [24]. Schrijver [20] was the first to give an interesting lower bound for the permanent in the form \( \text{per}(\overline{A}) \geq \prod_{1 \leq i < j \leq n} (1 - a_{i,j}) \), where \( \overline{A} \) is the matrix whose \((i, j)\)th entry is \( a_{i,j}(1 - a_{i,j}) \). Gurvits [12] has the current best lower bound, extending the ideas of [20]:

\[
\text{per}(A) \geq \prod_{1 \leq i < j \leq n} (1 - a_{i,j})^{1-a_{i,j}} . \tag{13}
\]

Let us see what \( L_4 \) gives, provided Conjecture 2 holds. Interpret \( X_i \) as selecting an entry \( j \) from row \( i \) of the doubly stochastic matrix. From this perspective, \( B_i \) is the event that, for some row \( j \neq i \), the random variables \( X_i \) and \( X_j \) selected entries belonging to the same column for rows \( i \) and \( j \). The product of \( n \) entries (one selected from each row) contributes to the permanent precisely when the chosen columns satisfy the event \( \bigwedge_{i=1}^n B_i \). Thus, \( \text{per}(A) = \Pr \left( \bigwedge_{i=1}^n B_i \right) \geq \prod_{i=1}^n (1 - \Pr(B_i)) = \prod_{i=1}^n \Pr(B_i) \). Now, \( B_i \) is the event that, for all \( k \neq i \), the value of \( X_i \) differs from the value of \( X_k \). Letting \( X_i = j \), the probability that \( X_k \neq j \) is \( 1 - a_{k,j} \) since the row sum is 1. Summing over \( j \), we have

\[
\Pr(B_i) = \sum_{j=1}^n a_{i,j} \prod_{k=1}^n (1 - a_{k,j}) . \tag{14}
\]

Finally, \( L_4 \) would give the lower bound

\[
\text{per}(A) \geq \prod_{i=1}^n \sum_{j=1}^n a_{i,j} \prod_{k=1}^n (1 - a_{k,j}) . \tag{15}
\]

It is interesting to compare our conjectured bound with the bound in [12]. [12] conjectures (13) makes the least fraction of the permanent of a doubly stochastic matrix on the matrix \( C \), in which \( c_{ii} = 1/2 \) and for \( i \neq j \) \( c_{ij} = 1/(2n - 2) \), with value \((\sqrt{2} + o(1))^{-n}\). Note that (13) evaluates to \((\sqrt{2}e + o(1))^{-n}\), while (15) evaluates to \((2\sqrt{e} + o(1))^{-n}\) on \( C \), so our lower bound performs worse. However, (15) has terms more similar to the permanent than the terms in (13), possibly making easier to estimate the performance of the approximation.

Another interesting matrix to compare the bounds is \( \frac{1}{n}J \), in which every entry is \( 1/n \). The famous van der Waerden conjecture stated that the permanent of non-negative doubly stochastic matrices is minimized on \( J \), with \( \text{per}(\frac{1}{n}J) = n!/n^n = (1 + o(1))e^{-n} \). This conjecture was proved by Friedland [11] with \( o(e^{-n}) \) error term and exactly by Falikman [9] and Egorychev [6]. It is easy to see that both (13) and (15) evaluate \((1 + o(1))e^{-n}\) on \( \frac{1}{n}J \).

Evidence for the validity of Conjecture 2 is that it holds for \( A = \frac{1}{n}J \) or \( A \) is a permutation matrix. For any fixed event \( B_i \) and any subset \( S = \{k_1, k_2, \ldots, k_s\} \) of the vertices (with \( i \not\in S \), we have this generalization of (14):

\[
\Pr \left( \bigwedge_{j \in S} B_j \right) = \sum_{T \subseteq [n]} \sum_{|T| = s \text{ injection}} b_{k_1 \pi(k_1)} b_{k_2 \pi(k_2)} \cdots b_{k_s \pi(k_s)} \prod_{t \not\in S} (1 - \sum_{t \in T} b_{tt}) . \tag{16}
\]
Using (16), for \( A = \frac{1}{n} J \), the condition (4) boils down to

\[
Pr(B_i) = \left(1 - \frac{1}{n}\right)^{n-1} \leq \left(1 - \frac{1}{n-s}\right)^{n-s-1}
\]

\[
= \frac{(n+s+1)^{n-s-1}}{(n)^{n-s}(1 - \frac{s+1}{n})^{n-s-1}} = Pr\left(\bigwedge_{j \in S \cup \{i\}} B_j \bigg| \bigwedge_{j \in S} B_j\right).
\]

Proving this inequality for arbitrary doubly stochastic \( A \) has so far eluded us. An alternative proof to Conjecture 2 with \( A = \frac{1}{n} J \) is using Theorem 7 with \( B_i \) instead of \( A_i \). Theorem 7 was designed for this in [16], although the context was estimating the number of injections.

**Acknowledgment** This research was supported in part by the NSF DMS contract 1000475. The third author acknowledges financial support from grant #FA9550-12-1-0405 from the U.S. Air Force Office of Scientific Research (AFOSR) and the Defense Advanced Research Projects Agency (DARPA). We thank Eva Czabarka for her useful suggestions to the manuscript.

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