Quest for Negative Dependency Graphs

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Lopsided Lovász Local Lemma

Negative Dependency Digraph G

- Each vertex i corresponds to an event A_i.
- For each event A_i and any subset S of its non-neighbors in G,

$$\Pr\left(\bigwedge_{j\in[k]} \overline{A_j}\right) \neq 0 \rightarrow \Pr\left(A_i \,\middle|\, \bigwedge_{j\in[k]} \overline{A_j}\right) \leq \Pr\left(A_i\right).$$

Lopsided Lovász Local Lemma

Lemma (Lopsided Lovász Local Lemma, Erdős-Spencer)

Let $\{A_i \mid i \in [n]\}$ *be a collection of events having a negative dependency digraph* G.
If there are real numbers $x_i \in [0, 1)$ *such that, for all* i,

$$\Pr(A_i) \leq x_i \prod_{(i,j) \in E(G)} (1 - x_j),$$

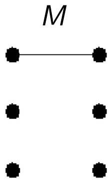
then

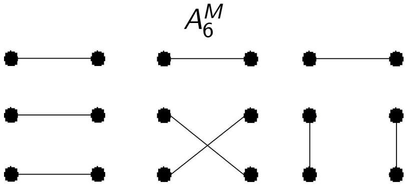
$$\Pr\left(\bigwedge_{i=1}^{n} \overline{A_i}\right) > 0.$$

Random Matchings in K_N

The canonical event for the matching M is

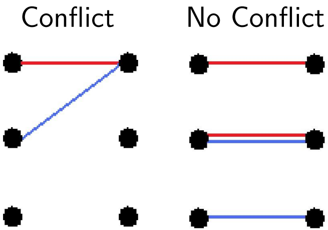
$$A_N^M = \{\text{perfect matchings } M' \text{ of } K_N \ (N \text{ even}) \mid M \subseteq M'\}.$$





Random Matchings in K_N

Two matchings M_1 and M_2 conflict provided there are edges $e \in M_1$ and $f \in M_2$ such that $|e \cap f| = 1$.



The conflict graph G for a collection of events $\{A_i \mid i \in [n]\}$ is the graph on $[n]$ with

$$E(G) = \{ij \mid A_i \text{ and } A_j \text{ conflict}\}.$$

Random Matchings in K_N

Theorem (Lu-Székely)

Let $\mathcal{M}$ be a collection of matchings in K_N. The conflict graph for the collection of canonical events $\{A^M \mid M \in \mathcal{M}\}$ is a negative dependency graph.

Some work allows

- "uniform hypergraph matchings" instead of "graph matchings",
- "r-matching" instead of "perfect matching", and
- "complete bipartite" instead of "complete".

Classification of Matching Hosts

Call a graph G good provided

- G is connected,
- for every edge e of G, the canonical event A^e is nonempty, and
- given any collection $\mathcal{M}$ of matchings in G, the conflict graph for the collection of canonical events $\{A^M \mid M \in \mathcal{M}\}$ is a negative dependency graph.

Proposition (M.)

The only good graphs of order $2n$ are $K_{n,n}$ and K_{2n}.

Classification of Matching Hosts

Lemma

If there are partial matchings M_1 and M_2 of a graph G such that

- M_1 *and* M_2 *do not conflict,*
- $A^{M_1} \subseteq \overline{A^{M_2}}$, *and*
- $A^{M_1}, A^{M_2} \neq \emptyset$,

then G is bad.

Classification of Matching Hosts

Proof.

$$\Pr\left(A^{M_1} \,\middle|\, \overline{A^{M_2}}\right) = \frac{\left|A^{M_1} \wedge \overline{A^{M_2}}\right|}{\left|\overline{A^{M_2}}\right|} = \frac{\left|A^{M_1}\right|}{\left|\overline{A^{M_2}}\right|} > \frac{\left|A^{M_1}\right|}{\left|\Omega\right|} = \Pr\left(A^{M_1}\right)$$

The inequality uses the assumptions that $|A^{M_1}| \neq 0$ and $|\overline{A^{M_2}}| < |\Omega|$ (since $A^{M_2} \neq \emptyset$). □

Classification of Matching Hosts

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The inequality uses the assumptions that $\left|A^{M_1}\right| \neq 0$ and $\left|\overline{A^{M_2}}\right| < |\Omega|$ (since $A^{M_2} \neq \emptyset$). □

Classification of Matching Hosts

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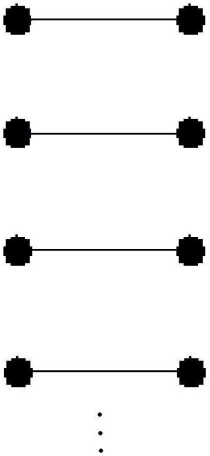
Classification of Matching Hosts

Every good graph of order $2n$ contains a $K_{n,n}$ subgraph.



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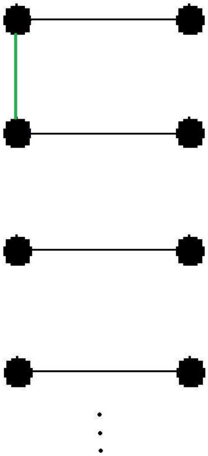

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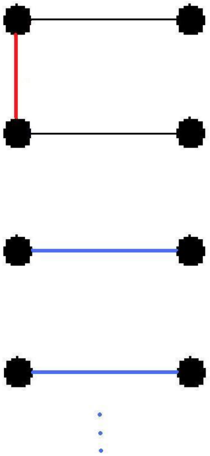

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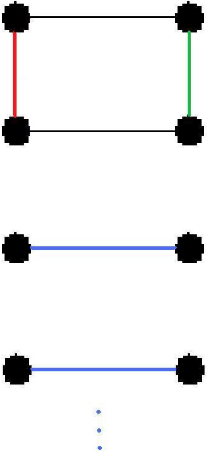

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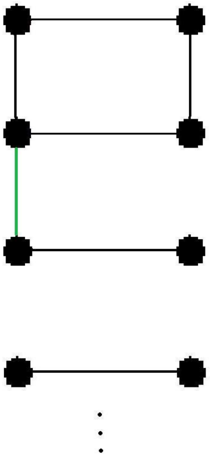

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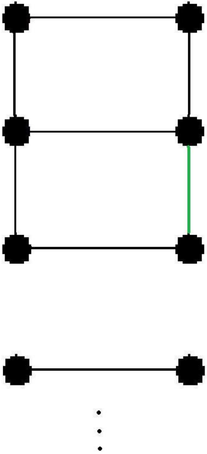

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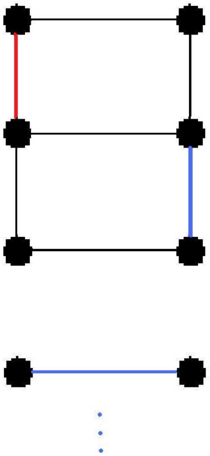

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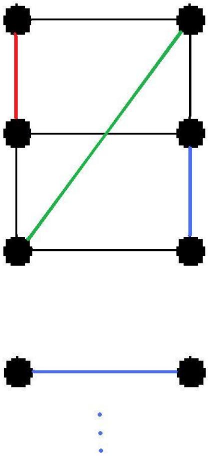

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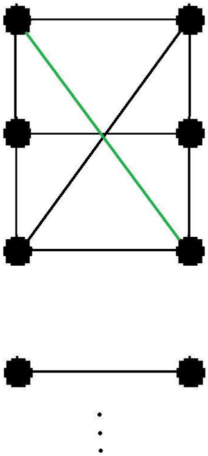
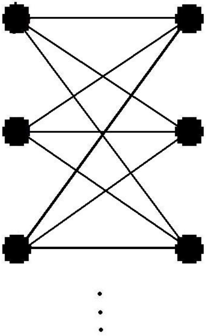


Classification of Matching Hosts

Every good graph of order $2n$ is either $K_{n,n}$ or K_{2n}.



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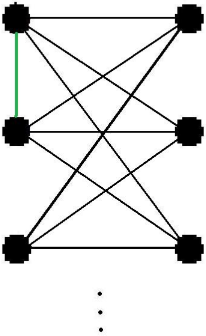

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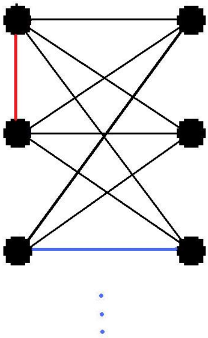

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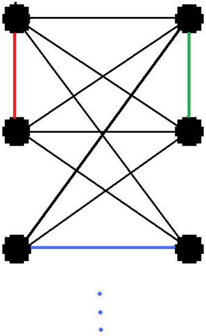

. . . and so on to K_{2n}.

Classification of Matching Hosts

What next?

- Remove connectivity requirement.
- Remove requirement that A^e nonempty for all e.
- Extend to uniform hypergraphs.

Random Partitions

A partial partition of $[n]$ is a collection of disjoint subsets of $[n]$. We say two partial partitions conflict whenever their union is not a partial partition.

Conflict	No Conflict
$A = \{\{1,2\}\}$	$A = \{\{1\},\{2\}\}$
$B = \{\{1,3\}\}$	$B = \{\{2\},\{3\}\}$

Given a partial partition P, define the canonical event

$$A^P = \{\text{partitions } P' \text{ of } [n] \mid P \subseteq P'\}.$$

Random Partitions

Conjecture

Let $\mathcal{P}$ be a collection of partial partitions of an N-element set. The conflict graph for the collection of canonical events $\{A^P \mid P \in \mathcal{P}\}$ is a negative dependency graph.

Random Partitions

We are given partial partitions P, P_1, ..., and P_k such that P does not conflict with any of the P_i. We want to show

$$\Pr\left(A^P \,\middle|\, \bigwedge_{i=1}^{k} \overline{A^{P_i}}\right) \leq \Pr\left(A^P\right),$$

which is equivalent to

$$\Pr\left(A^P\right)\Pr\left(\bigvee_{i=1}^{k} A^{P_i}\right) \leq \Pr\left(A^P \wedge \left(\bigvee_{i=1}^{k} A^{P_i}\right)\right),$$

which can be rewritten as

$$\left|A^P\right| \left|\bigcup_{i=1}^{k} A^{P_i}\right| \leq B_N \left|A^P \cap \left(\bigcup_{i=1}^{k} A^{P_i}\right)\right|.$$

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Further assume the partial partitions P, P_1, ..., P_k

- have disjoint underlying sets and
- each cover exactly m elements.

The inequality

$$\left|A^P\right|\left|\bigcup_{i=1}^{k} A^{P_i}\right| \leq B_N \left|A^P \cap \left(\bigcup_{i=1}^{k} A^{P_i}\right)\right|$$

becomes

$$B_{N-m}\sum_{i=1}^{k}(-1)^{i+1}\binom{k}{i}B_{N-i\cdot m} \leq B_N \sum_{i=1}^{k}(-1)^{i+1}\binom{k}{i}B_{N-(i+1)\cdot m}$$

by inclusion-exclusion.

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by inclusion-exclusion.

Random Spanning Trees

Two forests F_1 and F_2 conflict if they are not vertex disjoint. Given a forest F in K_N, define the canonical event

$$A^F = \{\text{spanning trees } T \text{ of } K_N \mid F \subseteq T\}.$$

Random Spanning Trees

Theorem (Székely)

Let $\mathcal{F}$ be a collection of forests in K_N. The conflict graph for the collection of canonical events $\{A^F \mid F \in \mathcal{F}\}$ is a negative dependency graph.

Random Spanning Trees

We are given forests F, F_1, ..., and F_k such that F does not conflict with any of the F_i. We want to show

$$\Pr\left(A^F \;\middle|\; \bigwedge_{i=1}^{k} \overline{A^{F_i}}\right) \leq \Pr\left(A^F\right),$$

which is equivalent to

$$\Pr\left(A^F\right)\Pr\left(\bigvee_{i=1}^{k} A^{F_i}\right) \leq \Pr\left(A^F \wedge \left(\bigvee_{i=1}^{k} A^{F_i}\right)\right).$$

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By inclusion-exclusion,

$$\Pr\left(\bigvee_{i\in[k]} A^{F_i}\right) = \sum_{R\subseteq[k],|R|\geq 1} \Pr\left(\bigwedge_{i\in R} A^{F_i}\right)(-1)^{|R|-1}$$

and

$$\Pr\left(A^F \wedge \left(\bigvee_{i\in[k]} A^{F_i}\right)\right) = \sum_{R\subseteq[k],|R|\geq 1} \Pr\left(A^F \wedge \left(\bigwedge_{i\in R} A^{F_i}\right)\right)(-1)^{|R|-1}.$$

Random Spanning Trees

The desired inequality can now be rewritten

$$\sum_{R \subseteq [k], |R| \geq 1} \Pr\left(A^F\right) \Pr\left(\bigwedge_{i \in R} A^{F_i}\right) (-1)^{|R|-1}$$

$$\leq \sum_{R \subseteq [k], |R| \geq 1} \Pr\left(A^F \wedge \left(\bigwedge_{i \in R} A^{F_i}\right)\right) (-1)^{|R|-1}.$$

We conclude by showing

$$\Pr\left(A^F\right)\Pr\left(\bigwedge_{i\in R} A^{F_j}\right) = \Pr\left(A^F \wedge \left(\bigwedge_{i\in R} A^{F_j}\right)\right).$$

If $\bigcup_{i\in R} F_i$ contains a cycle, then both sides are zero.
Otherwise, $\bigcup_{i\in R} F_i$ is a forest disjoint from F, so the corresponding events are independent (details omitted). □

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Random Spanning Trees

Two forests F_1 and F_2 weakly conflict if there are edges $e \in F_1$ and $f \in F_2$ such that $|e \cap f| = 1$.

Theorem (Székely)

Let $\mathcal{F}$ be a collection of forests in K_N. The weak conflict graph for the collection of canonical events $\{A^F \mid F \in \mathcal{F}\}$ is a negative dependency graph.

Random Spanning Trees

We are given forests F, F_1, ..., and F_k such that F does not weakly conflict with any of the F_i. We want to show

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Random Spanning Trees

$$\Pr\left(A^F \wedge \left(\bigwedge_{i=1}^{k} \overline{A^{F_i}}\right)\right) = \Pr\left(A^F \wedge \left(\bigwedge_{i=1}^{k} \overline{A^{F_i \setminus F}}\right)\right)$$

$$= \Pr\left(A^F\right) \Pr\left(\bigwedge_{i=1}^{k} \overline{A^{F_i \setminus F}}\right)$$

$$\leq \Pr\left(A^F\right) \Pr\left(\bigwedge_{i=1}^{k} \overline{A^{F_i}}\right)$$

$\square$

Random Spanning Trees

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\end{aligned}
$$

□

Permanent of Doubly-Stochastic Matrix

A doubly stochastic matrix is in one which every row and column sum is equal to 1.

The permanent of an $n \times n$ matrix A is

$$\text{per}(A) = \sum_{\sigma \in S_n} \prod_{i=1}^{n} a_{i,\sigma(i)}.$$

A typical term in the summation:

$$\begin{pmatrix} \frac{1}{4} & \frac{1}{4} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{4} & \frac{1}{4} \\ \frac{1}{4} & \frac{1}{2} & \frac{1}{4} \end{pmatrix}$$

Permanent of Doubly-Stochastic Matrix

For each row i, define the random variable $X_i = j$ ("select column j for row i") with probability $a_{i,j}$.

$$\begin{pmatrix} \frac{1}{4} & \color{red}{\frac{1}{4}} & \frac{1}{2} \\ \color{red}{\frac{1}{2}} & \frac{1}{4} & \frac{1}{4} \\ \frac{1}{4} & \frac{1}{2} & \color{red}{\frac{1}{4}} \end{pmatrix}$$

$$\Pr(X_1 = 2) = \frac{1}{4}$$
$$\Pr(X_2 = 1) = \frac{1}{2}$$
$$\Pr(X_3 = 3) = \frac{1}{4}$$

Permanent of Doubly-Stochastic Matrix

For each random variable i, define the event B_i to mean $X_i = X_j$ for some $i \neq j$.
Notice,

$$\text{per}(A) = \Pr\left(\bigwedge_{i=1}^{n} \overline{B_i}\right) \geq \prod_{i=1}^{n} \Pr\left(\overline{B_i}\right),$$

so the Lovász Local Lemma could give a lower bound.

Conjecture

The collection of events $\{B_i \mid 1 \leq i \leq n\}$ are the vertices of an edgeless negative dependency graph.

Permanent of Doubly-Stochastic Matrix

Unfortunately, there is a probabilistic counterexample (Gurvits, 2012+).

What next?

- Find an explicit counterexample.
- Direct verification for small dimension.
- Redefine the events B_i appropriately.

Thanks

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Download This Talk
AustinMohr.com

Further Details
L. Lu, A. Mohr, and L.A. Székely
Quest for Negative Dependency Graphs