The Probabilistic Method

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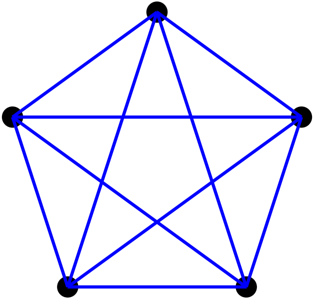
Department of Mathematics
Nebraska Wesleyan University

March 14, 2014

Poorly Balanced Parties

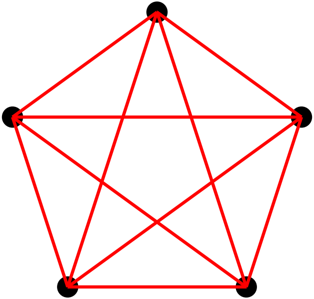
Too Familiar



Acquaintances

Poorly Balanced Parties

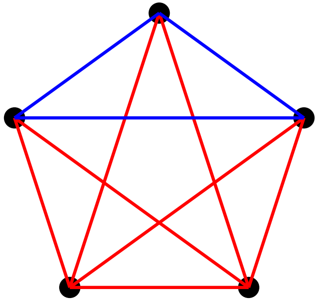
Too Awkward



Strangers

Poorly Balanced Parties

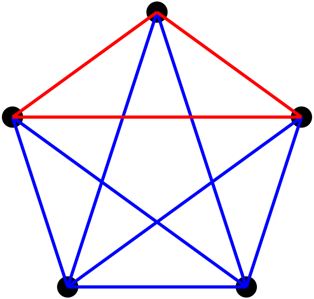
Too Cliquey



Acquaintances Strangers

Poorly Balanced Parties

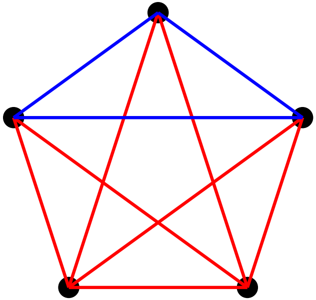
Too Stressful



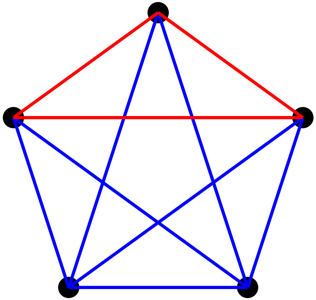
Acquaintances Strangers

My Ideal Party

My ideal party has neither three mutual acquaintances nor three mutual strangers.



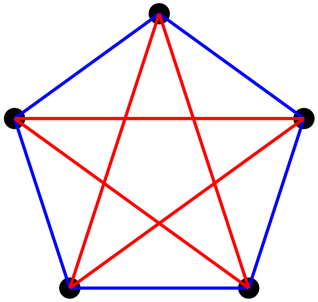
three mutual acquaintances



three mutual strangers

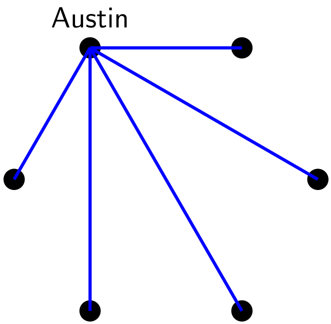
My Ideal Party

An ideal party of five.



No Ideal Party of Six

Focus on just one guest.



One of the following is always true:

- There are at least three people I know.
- There are at least three people I don't know.

No Ideal Party of Six

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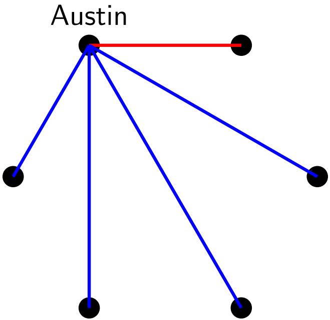

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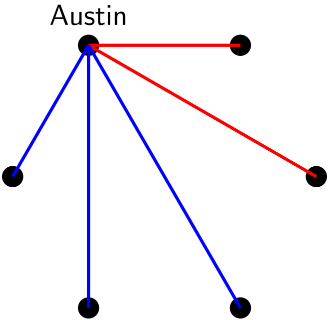

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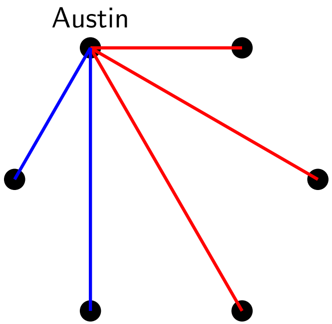

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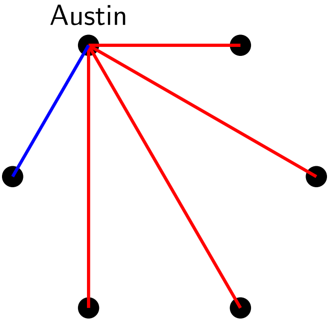

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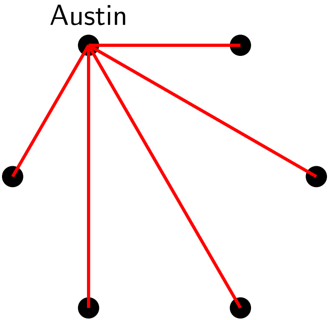

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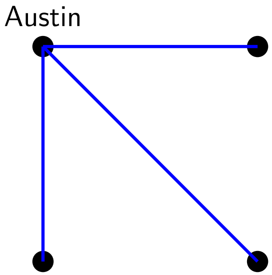
No Ideal Party of Six

If there are three people I know...



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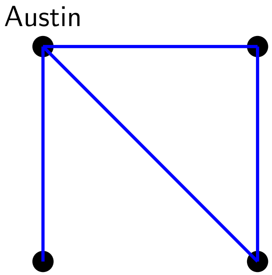

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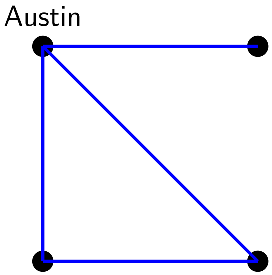

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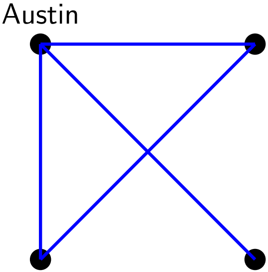

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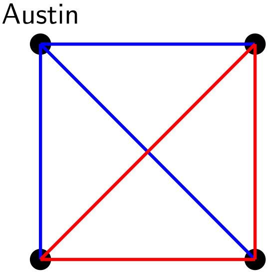
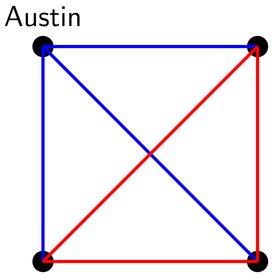


No Ideal Party of Six

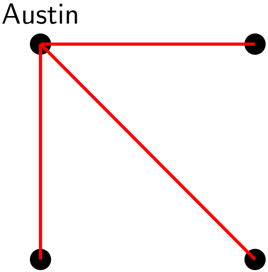
If there are three people I know...



...there is no ideal party.

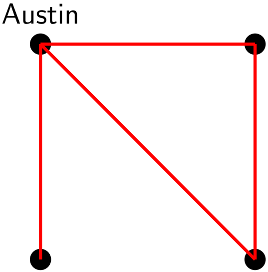
No Ideal Party of Six

If there are three people I don't know...



No Ideal Party of Six

If there are three people I don't know...



No Ideal Party of Six

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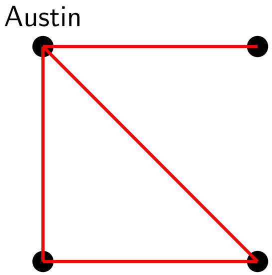

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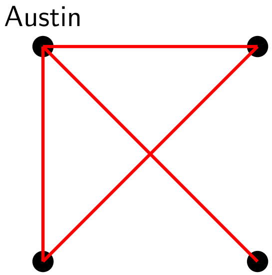

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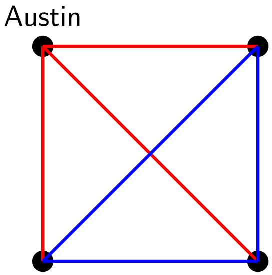

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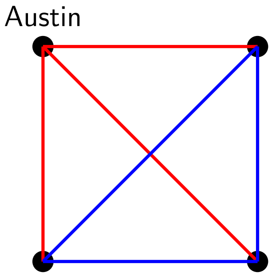
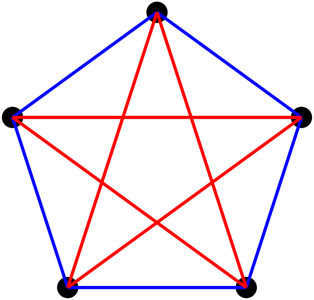


...there is no ideal party.

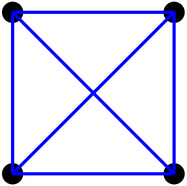
No Ideal Party of Six

The largest ideal party consists of only five people.

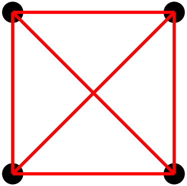


My Slightly Less Ideal Party

My slightly less ideal party has neither **four** mutual acquaintances nor **four** mutual strangers.

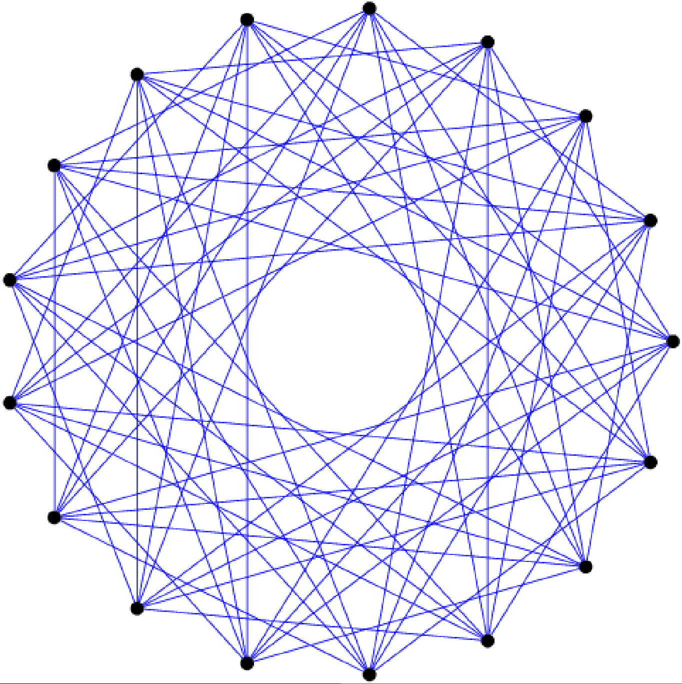


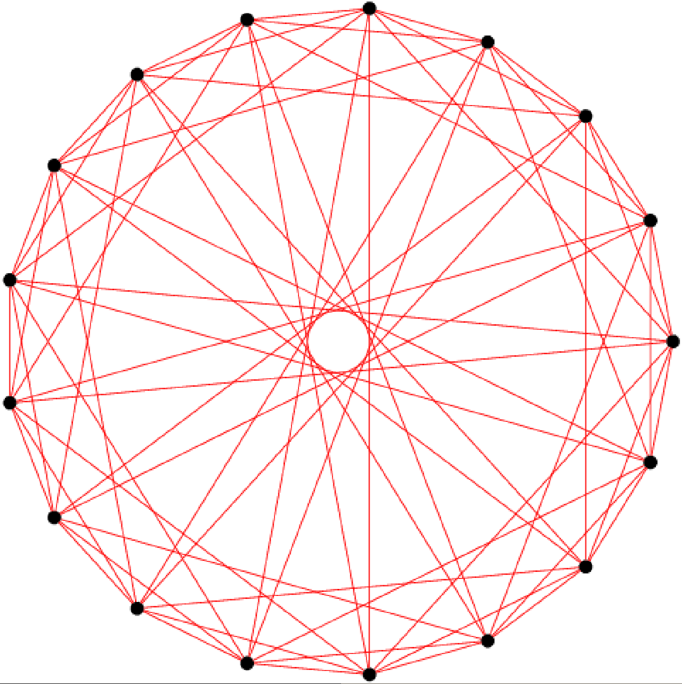
four mutual acquaintances



four mutual strangers

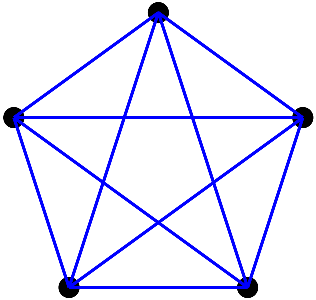
The largest such party consists of only seventeen people.



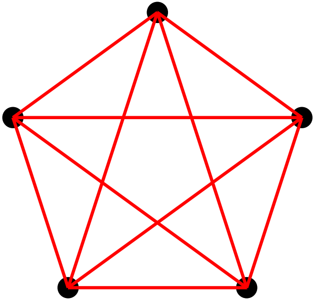


My Much Less Ideal Party

My much less ideal party has neither **five** mutual acquaintances nor **five** mutual strangers.



five mutual acquaintances



red mutual strangers

- There is such a party with 42 guests.
- There is no such party with 49 guests.

Ramsey's Theorem

Theorem

Your party will not be ideal if you invite too many people.

Frank P. Ramsey

Ramsey's Theorem

Theorem

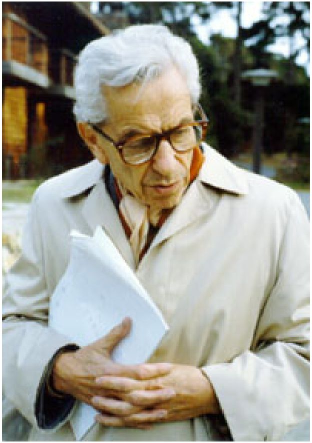
For any natural number k, there is a (large) natural number n such that any party of n people will contain either k mutual acquaintances or k mutual strangers.

Frank P. Ramsey

Discovering Large Ideal Parties

The Probabilistic Method

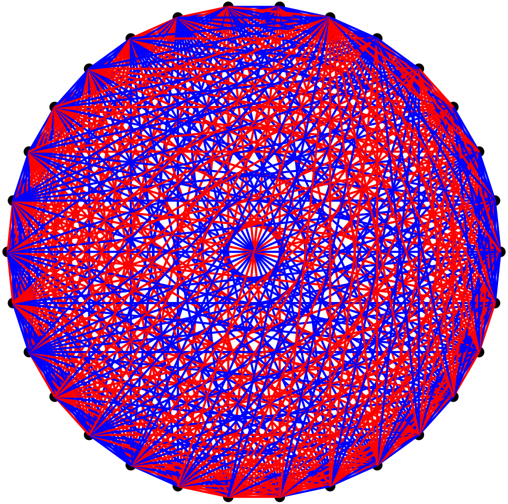
- Approach your task randomly.
- Show that the probability of a desirable outcome is nonzero.



Paul Erdős

Discovering Large Ideal Parties

Flip a coin for each connection.
Color blue on heads and red on tails.



Discovering Large Ideal Parties

For any group of k people, the probability they are mutual acquaintances or mutual strangers is

$$\frac{2}{2^k} = 2^{1-\binom{k}{2}}.$$

If there are n people in total at the party, then there are $\binom{n}{k}$ ways to select a group of k people.

Therefore, the probability that the party contains k mutual acquaintances or k mutual strangers is at most

$$\binom{n}{k} 2^{1-\binom{k}{2}}.$$

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Discovering Large Ideal Parties

If n is chosen small enough to satisfy

$$\binom{n}{k} 2^{1-k} < 1,$$

then there is nonzero probability that the party contains **neither** k mutual acquaintances nor k mutual strangers.

What's a nice large value to choose for n?

$$\binom{n}{k} = \frac{n!}{(n-k)!k!} < \frac{n^k}{k!}$$

$$2^{1-\binom{k}{2}} = \frac{2^{1+\frac{k}{2}}}{2^{\frac{k^2}{2}}}$$

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Discovering Large Ideal Parties

We have

$$\binom{n}{k}2^{1-k} < \frac{n^k}{k!}\frac{2^{1+\frac{k}{2}}}{2^{\frac{k^2}{2}}},$$

which is less than 1 when

$$n < \left(\frac{k!}{2}\right)^{\frac{1}{k}}\frac{2^{\frac{k}{2}}}{\sqrt{2}}.$$

Discovering Large Ideal Parties

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Discovering Large Ideal Parties

Using Stirling's approximation

$$k! \geq \frac{\sqrt{2\pi}k^{k+\frac{1}{2}}}{e^k},$$

we write

$$\begin{aligned}\left(\frac{k!}{2}\right)^{\frac{1}{k}} &\geq \left(\frac{\sqrt{2\pi}k^{k+\frac{1}{2}}}{e^k}\right)^{\frac{1}{k}} \\ &= \frac{k}{e}\left(\sqrt{2\pi}k^{\frac{1}{2}}\right)^{\frac{1}{k}} \\ &> \frac{k}{e}.\end{aligned}$$

Discovering Large Ideal Parties

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Discovering Large Ideal Parties

If

$$n < \frac{1}{\sqrt{2}e} k 2^{\frac{k}{2}},$$

then the probability that a random party contains k mutual acquaintances or k mutual strangers is less than 1.

In other words, the probability that a random party contains **neither** of these groups is greater than 0, and therefore such a party exists.

Discovering Large Ideal Parties

Theorem (Erdős, 1947)

For any natural number k, there is a party of $\frac{1}{\sqrt{2}e}k2^{\frac{k}{2}}$ people that contains neither k mutual acquaintances nor k mutual strangers.

In seventy years, the result has only been improved by a factor of 2.

Theorem (Spencer, 1975)

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Discovering Large Ideal Parties

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For any natural number k, there is a party of $\frac{\sqrt{2}}{e}k2^{\frac{k}{2}}$ people that contains neither k mutual acquaintances nor k mutual strangers.

Discovering Large Ideal Parties

Spencer's result says there is a party of 166 people that contains neither 10 mutual acquaintances nor 10 mutual strangers.

It is known that there is such a party with 798 people.

Discovering Large Ideal Parties

Spencer's result says there is a party of **166** people that contains neither 10 mutual acquaintances nor 10 mutual strangers.

It is known that there is such a party with **798** people.

Thanks

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Search the web for "Ramsey Numbers" to learn more.