Math 711 Homework 7

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Problem 1

Let $\{X_n, n \geq 1\}$ be independent random variables with $\mathbb{E}(X_n) = 0$ and $\mathbb{V}(X_n) = \sigma_n^2 < \infty$. Let $s_n^2 = \sum_{i=1}^n \sigma_i^2$.

Proposition 1. *If there exists a* $\delta > 0$ *such that, as* $n \to \infty$,

$$\frac{\sum_{i=1}^n \mathbb{E}|X_i|^{2+\delta}}{s_n^{2+\delta}} \to 0,$$

then the Lindeberg condition below holds:

$$\forall \epsilon > 0 : \lim_{n\to\infty} \frac{1}{s_n^2} \sum_{i=1}^n \mathbb{E}\left[X_i^2 I\{|X_i| > \epsilon s_n\}\right] \to 0.$$

Proof. Let $\epsilon > 0$ be given. It follows directly that

$$\begin{aligned}
\frac{1}{s_n^2} \sum_{i=1}^n \mathbb{E}\left(X_i^2 I\{|X_i| > \epsilon s_n\}\right) &= \sum_{i=1}^n \mathbb{E}\left(\left|\frac{X_i}{s_n}\right|^2 I\{|X_i| > \epsilon s_n\}\right) \\
&\leq \sum_{i=1}^n \mathbb{E}\left(\left|\frac{X_i}{s_n}\right|^2 \left|\frac{X_i}{\epsilon s_n}\right|^\delta I\{|X_i| > \epsilon s_n\}\right) \\
&\leq \frac{1}{\epsilon^\delta} \frac{\sum_{i=1}^n \mathbb{E}|X_i|^{2+\delta}}{s_n^{2+\delta}} \\
&\to 0.
\end{aligned}$$

□

Remark 1. *Define* $S_n = \sum_{i=1}^n X_i$. *Since the Lindeberg condition holds for* $\{X_n\}$, *the Lindeberg-Feller central limit theorem implies that* $\frac{S_n}{s_n} \Rightarrow N(0,1)$.

Problem 2

Let $\{X_n, n \geq 1\}$ be a sequence of independent random variables with mean zero and variance σ_n^2.

Proposition 2. *Even if there exists a $B > 0$ such that $\forall n : \frac{1}{B} \leq \sigma_n^2 \leq B$, it does* **not** *follow that*

$$\frac{\sum_{i=1}^{n} X_i}{\sqrt{\sum_{i=1}^{n} \sigma_i^2}} \Rightarrow N(0,1). \tag{1}$$

Proof. For a counterexample, define $X_n = nI([0, \frac{1}{n^2}]) - nI([1 - \frac{1}{n^2}, 1])$. We have, for all n,

$$\begin{aligned} \mathbb{E}(X_n) &= n \cdot \frac{1}{n^2} - n \cdot \frac{1}{n^2} \\ &= 0 \end{aligned}$$

and

$$\begin{aligned} \mathbb{V}(X_n) &= \mathbb{E}\left(X_n^2\right) + \mathbb{E}(X_n)^2 \\ &= n^2 \cdot \frac{1}{n^2} + n^2 \cdot \frac{1}{n^2} \\ &= 2. \end{aligned}$$

Let now $Y_n = \frac{\sum_{k=1}^{n} X_k}{\sqrt{\sum_{k=1}^{n} \sigma_k^2}}$. For all n, Y_n is equal to zero on the interval $[\frac{1}{4}, \frac{3}{4}]$. Thus, the distribution of the limiting random variable is constant on this interval, and so in particular is not normal.

□

Proposition 3. *If, in addition, for some $B < \infty$, $\forall n : P\{|X_n| \leq B\} = 1$, ie., the rvs are bounded, then (1) holds.*

Proof. We establish the condition in Problem 1 and then appeal to the Lindeberg-Feller central limit theorem.

We have

$$\begin{aligned} \frac{\sum_{i=1}^{n} \mathbb{E}\left(|X_i|^{2+\delta}\right)}{s_n^{2+\delta}} &\leq \frac{\sum_{i=1}^{n} \mathbb{E}\left(B^{2+\delta}\right)}{\left(\frac{n}{B}\right)^{2+\delta}} \\ &= \frac{nB^{2+\delta}}{\frac{n^{2+\delta}}{B^{2+\delta}}} \\ &\to 0. \end{aligned}$$

□

Problem 3

Suppose that Y_s has a Poisson distribution with parameter s, not necessarily an integer, so that

$$P\{Y_s = k\} = \frac{\exp(-s)s^k}{k!}, k = 0, 1, 2, \ldots.$$

Proposition 4. *As* $s \to \infty$,

$$\frac{Y_s - s}{\sqrt{s}} \Rightarrow N(0,1).$$

Proof. Let $Z_s = \frac{Y_s - s}{\sqrt{s}}$. By the Uniqueness Theorem, it suffices to show that the characteristic function of Z_s converges to the characteristic function of a standard normal random variable.

We have

$$\begin{aligned} \phi_{Z_s}(t) &= \phi_{Y_s}\left(\frac{t}{\sqrt{s}}\right)\exp(-it\sqrt{s}) \\ &= \exp\left(s(e^{\frac{it}{\sqrt{s}}} - 1)\right)\exp(-it\sqrt{s}) \\ &= \exp\left(se^{\frac{it}{\sqrt{s}}} - s - it\sqrt{s}\right). \end{aligned}$$

Taking logarithms, we proceed by showing the exponent converges to $-\frac{t^2}{2}$.

$$\begin{aligned} se^{\frac{it}{\sqrt{s}}} - s - it\sqrt{s} &= s\sum_{k=0}^{\infty}\frac{(it)^k}{(\sqrt{s})^k k!} - s - it\sqrt{s} \\ &= s\left(1 + \frac{it}{\sqrt{s}} - \frac{t^2}{2s} - \frac{it^3}{6s^{\frac{3}{2}}} + \cdots\right) - s - it\sqrt{s} \\ &= \left(s + it\sqrt{s} - \frac{t^2}{2} - \frac{it^3}{6s^{\frac{1}{2}}} + \cdots\right) - s - it\sqrt{s} \\ &= -\frac{t^2}{2} - \frac{it^3}{6s^{\frac{1}{2}}} + \cdots. \end{aligned}$$

Since t is constant, the above converges to $-\frac{t^2}{2}$ as $s \to \infty$, as desired. $\square$

Problem 4

Let $\{X_n, n \geq 1\}$ be independent random variables with $X_n \sim N(0, \sigma_n^2)$. Let $s_n^2 = \sum_{i=1}^n \sigma_i^2$ and $S_n = \sum_{i=1}^n X_i$.

Proposition 5. *If the* σ_n^2 *are chosen such that* $\max_{i\leq n}\sigma_i^2/s_n^2$ *does not converge to zero as* $n \to \infty$*, then the Lindeberg condition does not hold.*

Proof. Define $\sigma_k = 2^{-(k-1)}$. We have, for all n,

$$\begin{aligned} \max_{i\leq n}\frac{\sigma_i^2}{s_n^2} &= \frac{1}{\sum_{k=1}^n 2^{-(k-1)}} \\ &= \frac{1}{2}, \end{aligned}$$

and so $\max_{i\leq n}\frac{\sigma_i^2}{s_n^2}$ does not converge to zero.

As for the Lindeberg condition, observe first that $s_n \to \frac{\sqrt{2}}{2}$. Choose ϵ_0 such that $P\{|X_1| > \epsilon_0 \frac{\sqrt{2}}{2}\} > 0$ (such ϵ_0 exists since X_1 is not identically zero). We have, for all n,

$$\begin{aligned}\sum_{i=1}^{n} \mathbb{E}\left[X_i^2 I\{|X_i| > \epsilon_0 s_n\}\right] &\geq \mathbb{E}\left[X_1^2 I\{|X_1| > \epsilon_0 s_n\}\right] \\ &\to \mathbb{E}\left[X_1^2 I\left\{|X_1| > \epsilon_0 \frac{\sqrt{2}}{2}\right\}\right].\end{aligned}$$

By our choice of ϵ_0, the expectation above is equal to some positive constant not depending on n. Therefore, the Lindeberg condition does not hold. ☐

Proposition 6. *For the choice of* σ_n^2 *in the proposition above, we have* $S_n/s_n \Rightarrow N(0,1)$. *[Remark: This shows that the Lindeberg condition is not necessary for the CLT to hold.]*

Proof. We know that $S_n \sim N(0, s_n^2)$, and so the characteristic function $\phi_{S_n}(t) = \exp\left(-\frac{1}{2}t^2 s_n^2\right)$. Let now $Y_n = \frac{S_n}{s_n}$. We have

$$\begin{aligned}\phi_{Y_n}(t) &= \phi_{S_n}\left(\frac{t}{s_n}\right) \\ &= \exp\left(-\frac{1}{2}\left(\frac{t}{s_n}\right)^2 s_n^2\right) \\ &= \exp\left(-\frac{t^2}{2}\right).\end{aligned}$$

Therefore, by the Uniqueness Theorem, $\frac{S_n}{s_n}$ converges to a standard normal random variable. ☐

Proposition 7. *For the choice of* σ_n^2 *in the above proposition, the following condition does not hold:*

$$\forall \epsilon > 0, \max_{i \leq n} P\{|X_i| > \epsilon s_n\} \to 0.$$

Proof. Choose ϵ_0 such that $P\{|X_1| > \epsilon_0 \frac{\sqrt{2}}{2}\} > 0$ (such ϵ_0 exists since X_1 is not identically zero). We have, for all n,

$$\begin{aligned}\max_{i \leq n} P\{|X_i| > \epsilon_0 s_n\} &\geq P\{|X_1| > \epsilon_0 s_n\} \\ &\to P\left\{|X_1| > \epsilon_0 \frac{\sqrt{2}}{2}\right\}.\end{aligned}$$

By our choice of ϵ_0, the probability above is equal to some positive constant not depending on n. Therefore, the condition does not hold. ☐