Math 711 Homework 1

Austin Mohr

January 27, 2011

Problem 1

Proposition 1. *Convergence in probability of a sequence of random variables does not imply almost sure convergence of the sequence.*

Proof. Let $([0,1], \mathfrak{B}([0,1]), \lambda)$ be the ambient probability space. Consider a process wherein, at the n^{th} stage, we choose (independently at random) a closed subinteral F_n of $[0,1]$ of length $\frac{1}{n}$.

Define now the random variable $X_n = I(F_n)$. Evidently, $X \xrightarrow{\text{pr}} 0$, since, for each n, X_n is nonzero only on the interval of length $\frac{1}{n}$. Thus, for any $\epsilon > 0$,

$$P(|X_n| > \epsilon) = \frac{1}{n} \to 0.$$

To see that X_n does not enjoy almost sure convergence to zero, observe that, for each $\omega \in [0,1]$,

$$\begin{aligned}\sum_{n=1}^{\infty} P(X_n(\omega) \neq 0) &= \sum_{n=1}^{\infty} \frac{1}{n} \\ &= \infty.\end{aligned}$$

Thus, by the Borel Zero-One Law, $P(X_n(\omega) \neq 0 \text{ i.o.}) = 1$, and so it cannot be that X_n converges almost surely to zero. □

Problem 2

Proposition 2. *For any sequence $\{X_n\}$ of random variables, there exists a sequence of constants $\{a_n\}$ such that $\frac{X_n}{a_n}$ converges almost surely to zero.*

Proof. Let $\epsilon > 0$ be given. Choose a_n such that $P(|X_n| > \epsilon) \leq \frac{1}{2^n}$. Observe that

$$\begin{aligned}\sum_{n=1}^{\infty} P(|X_n| > \epsilon) &\leq \sum_{n=1}^{\infty} \frac{1}{2^n} \\ &< \infty,\end{aligned}$$

and so the Borel-Cantelli Lemma gives that $P([|X_n| > \epsilon] \text{ i.o.}) = 0$. That is, $X_n \xrightarrow{\text{a.s.}} 0$. □

Problem 3

Proposition 3. *For a monotone sequence of random variables, convergence in probability implies almost sure convergence.*

Proof. Let X_n be a monotone sequence of random variables with $X_n \xrightarrow{\text{pr}} X$. It follows that, for all $\epsilon > 0$,

$$\begin{aligned}
0 &= \lim_{n\to\infty} P(|X_n - X| > \epsilon) \\
&= \lim_{N\to\infty} P\left(\bigcup_{n\geq N} [|X_n - X| > \epsilon]\right) \qquad \text{(by monotonicity of the } X_n\text{)} \\
&= P\left(\limsup_{n\to\infty} [|X_n - X| > \epsilon]\right) \\
&= P([|X_n - X| > \epsilon] \text{ i.o.}).
\end{aligned}$$

Therefore, $X_n \xrightarrow{\text{a.s.}} X$. □

Problem 4

Proposition 4. *Let $\{X_n\}$ be a sequence of random variables and define, for each n, $Y_n = X_n I\{|X_n| \leq a_n\}$. There exists a sequence $\{a_n\}$ of positive real numbers such that $P\{X_n \neq Y_n \text{ i.o.}\} = 0$.*

Proof. For each n, choose a_n such that $P(|X_n| > a_n) \leq \frac{1}{2^n}$. Thus,

$$\begin{aligned}
\sum_{n=1}^{\infty} P(X_n \neq Y_n) &= \sum_{n=1}^{\infty} P(|X_n| > a_n) \\
&\leq \sum_{n=1}^{\infty} \frac{1}{2^n} \\
&< \infty.
\end{aligned}$$

By the Borel-Cantelli Lemma, we conclude that $P([X_n \neq Y_n] \text{ i.o.}) = 0$. □

Problem 5

Proposition 5. *Suppose n points are chosen randomly on the unit circle. Define the random variable X_n to be the arc length of the largest arc not containing any of the chosen points. In this case, $X_n \to 0$ almost surely.*

Proof. Let $\epsilon > 0$ be given. We have that

$$\begin{aligned} P([X_n > \epsilon] \text{ i.o.}) &= P\left(\limsup_{n\to\infty}[X_n > \epsilon]\right) \\ &= \lim_{N\to\infty} P\left(\bigcup_{n\geq N}[X_n > \epsilon]\right) \\ &= \lim_{n\to\infty} P(X_n > \epsilon) \qquad \text{(by monotonicity of the } X_n\text{).} \end{aligned}$$

We bound this last probability by breaking the unit circle into $\frac{4\pi}{\epsilon}$ disjoint intervals of length $\frac{\epsilon}{2}$. Thus, $P(X_n \leq \epsilon)$ is no larger than the probability of having a point contained in a every interval. Thus,

$$\begin{aligned} P([X_n > \epsilon] \text{ i.o.}) &= \lim_{n\to\infty} P(X_n > \epsilon) \\ &\leq \lim_{n\to\infty} \frac{4\pi}{\epsilon}\left(\frac{2\pi - \frac{\epsilon}{2}}{2\pi}\right)^n \\ &= 0, \end{aligned}$$

and so $X_n \xrightarrow{\text{a.s.}} 0$. □

Problem 6

Proposition 6. *If, for all $a < b$,*

$$P\{[\{X_n < a\} \text{ } i.o.] \text{ and } [\{X_n > b\} \text{ } i.o.]\} = 0,$$

then $lim_{n\to\infty} X_n$ exists almost surely.

Proof. By taking complements, we have

$$P\{[X_n \geq a] \text{ or } [X_n \leq b]\} = 1,$$

for all sufficiently large n. If it is the case that the former holds for all a, then $\lim_{n\to\infty} X_n = \infty$ almost surely. Similarly, if the latter holds for all b, then $\lim_{n\to\infty} X_n = -\infty$ almost surely. Otherwise, there is some c so that $P(X_n \leq b) = 1$ for all $b \geq c$ and $P(X_n \leq b) = 0$ for all $b < c$. By keeping $b = c$ fixed and letting $a \uparrow b$, we see that $\lim_{n\to\infty} X_n = c$ almost surely. □

Problem 7

Proposition 7. *If $X_n \to 0$ in probability, then, for any $\alpha > 0$,*

$$\frac{|X_n|^\alpha}{1 + |X_n|^\alpha} \xrightarrow{pr} 0.$$

Proof. Let $\epsilon > 0$ be given. Choose N such that

$$P(|X_n| > \epsilon^{\frac{1}{\alpha}}) < \epsilon,$$

for all $n \geq N$. It follows that

$$P(|X_n|^\alpha > \epsilon) < \epsilon$$

for all $n \geq N$. Now,

$$\frac{|X_n|^\alpha}{1 + |X_n|^\alpha} \leq |X_n|.$$

Thus, for all $n \geq N$,

$$\begin{aligned} P\left(\frac{|X_n|^\alpha}{1 + |X_n|^\alpha} > \epsilon\right) &\leq P(|X_n|^\alpha > \epsilon) \\ &< \epsilon, \end{aligned}$$

and so $\frac{|X_n|^\alpha}{1+|X_n|^\alpha} \xrightarrow{\text{pr}} 0$. □

Problem 8

Proposition 8. *If, for some $\alpha > 0$,*

$$\frac{|X_n|^\alpha}{1 + |X_n|^\alpha} \xrightarrow{pr} 0,$$

then $X_n \xrightarrow{pr} 0$.

Proof. Let $\epsilon > 0$ be given. Choose N such that

$$P\left(\frac{|X_n|^\alpha}{1 + |X_n|^\alpha} > \epsilon\right) < \epsilon$$

for all $n \geq N$. Now,

$$\left(\frac{|X_n|}{1 + |X_n|}\right)^\alpha \leq \frac{|X_n|^\alpha}{1 + |X_n|^\alpha}.$$

Thus, for all $n \geq N$,

$$\begin{aligned} P\left(\left(\frac{|X_n|}{1 + |X_n|}\right)^\alpha > \epsilon\right) &\leq P\left(\frac{|X_n|^\alpha}{1 + |X_n|^\alpha}\right) \\ &< \epsilon, \end{aligned}$$

and so

$$P\left(\frac{|X_n|}{1 + |X_n|} > \epsilon^{\frac{1}{\alpha}}\right) < \epsilon,$$

from which it follows that $X_n \xrightarrow{\text{pr}} 0$. □

Problem 9

Proposition 9. *Let $\{X_n\}$ be a collection of independent random variables with*

$$P\{X_n = n^2\} = \frac{1}{n^2} \text{ and } P\{X_n = -1\} = 1 - \frac{1}{n^2}$$

for all n. In this case, $\sum_{i=1}^n X_i$ converges almost surely to $-\infty$ as $n \to \infty$.

Proof. Observe first that, by definition of the X_n, $P(X_n \in \{n^2, -1\}) = 1$. That is, except for a null set, X_n takes on only the values n^2 or -1.

Now, we have

$$\begin{aligned}\sum_{n=1}^{\infty} P(X_n = n^2) &= \sum_{n=1}^{\infty} \frac{1}{n^2} \\ &< \infty,\end{aligned}$$

and so $P([X_n = n^2] \text{ i.o.}) = 0$ by the Borel-Cantelli Lemma.

Similarly,

$$\begin{aligned}\sum_{n=1}^{\infty} P(X_n = -1) &= \sum_{n=1}^{\infty} 1 - \frac{1}{n^2} \\ &= \infty,\end{aligned}$$

and so $P([X_n = -1] \text{ i.o.}) = 1$ by the Borel Zero-One Law (note here that we require the independence of the X_n).

Thus, for almost all $\omega \in \Omega$, there exists N_ω such that $X_n(\omega) = -1$ for all $n \geq N_\omega$. It follows that

$$\begin{aligned}\lim_{n\to\infty} S_n(\omega) &= \lim_{n\to\infty} \sum_{i=1}^{n} X_i(\omega) \\ &= \sum_{i=1}^{N_\omega} X_i(\omega) + \sum_{i>N_\omega} X_i(\omega) \\ &\leq \sum_{i=1}^{N_\omega} n^2 + \sum_{i>N_\omega} -1 \\ &= -\infty,\end{aligned}$$

and so $S_n \xrightarrow{\text{a.s.}} -\infty$. $\square$