Problem Set 2

MATH 776, Fall 2009, Mohr

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1 Problem 1

For edges $e, e' \in G$, write $e \sim e'$ if either $e = e'$ or e and e' lie on some common cycle in G.

Proposition 1.1. *The relation $\sim$ is an equivalence relation on $E(G)$ whose equivalence classes are the edge sets of the non-trivial blocks of G.*

Proof. To show that $\sim$ is an equivalence relation on $E(G)$, we verify that it is reflexive, symmetric, and transitive.

Claim 1.2. *The relation $\sim$ is reflexive.*

Proof. By definition, $e \sim e'$ whenever $e = e'$, so $e \sim e$. □

Claim 1.3. *The relation $\sim$ is symmetric.*

Proof. Let $e, e' \in E(G)$ with $e \sim e'$. Either $e = e'$ or e and e' lie on the some common cycle in G. In either case, switching the roles of e and e' yields the same statement (either $e' = e$ or e' and e lie on some common cycle in G), and so $e' \sim e$. □

Claim 1.4. *The relation $\sim$ is transitive.*

Proof. Let $e, f, g \in E(G)$ be such that $e \sim f$ and $f \sim g$. If $e = f$, simply rewrite the second relation statement to read "$e \sim g$". Similarly, if $f = g$, then rewrite the first relation statement to read "$e \sim g$".
For the remaining case, let e and f lie on the cycle C in G. We also know that f and g lie on the same cycle in G. Hence, it must be that g lies on C, but this is precisely the same as saying $e \sim g$. □

Now, let $e \in E(G)$. We know that e belongs to a unique equivalence class A under $\sim$, and that e belongs to a unique block B in G. To show that the equivalence classes of $E(G)$ with respect to $\sim$ are precisely the non-trivial blocks of G, it suffices to show that $A = E(B)$.
($\subseteq$) Let $f \in A$. By definition, $e \sim f$, and so either $e = f$ or e and f lie on some common cycle in G. If $e = f$, then certainly e and f belong to the same block of G,

and so $f \in B$. Suppose now that e and f lie on some common cycle C in G. Now, since $e \in B$, the maximal 2-connected subgraph of G that contains C is in fact B. Hence, $f \in B$.
($\supseteq$) Let $f \in B$. Now, B is either a bridge or a maximal 2-connected subgraph of G. In the first case, we have that $|B| = 1$, and so it must be that $e = f$. Hence, $e \sim f$. In the second case, B is 2-connected, and so B can be constructed from some cycle by successively adding H-paths to graphs H already constructed (Dietel 3.1.3). Hence, we can find some cycle in B containing both e and f, and so $e \sim f$. ☐

2 Problem 2

Proposition 2.1. *Let G be a 3-connected graph, and let xy be an edge of G. The graph G/xy is 3-connected if and only if $G - \{x, y\}$ is 2-connected.*

Proof. ($\Rightarrow$) Let G/xy be 3-connected and suppose, for the purpose of contradiction, that $G - \{x, y\}$ is not 2-connected. That is, $G - \{x, y\}$ is either disconnected or 1-connected.
In the first case, we see that $\{x, y\}$ is a separating set for G, which is a contradiction with the fact that G is 3-connected.
In the second case, there exists a cutvertex z of $G - \{x, y\}$, and so $\{x, y, z\}$ is a separating set for G. Let v be the vertex resulting from the contraction of the edge xy. We see that $\{v, z\}$ is a separating set for G/xy, which is a contradiction with the fact that G/xy is 3-connected.
($\Leftarrow$) Let $G - \{x, y\}$ be 2-connected and suppose, for the purpose of contradiction, that G/xy is not 3-connected. That is, G/xy is either disconnected or is at most 2-connected.
In the first case, it must be that G itself is disconnected, as the contraction of an edge never disconnects a graph. This is contrary to the fact that G is 3-connected.
For the second case, suppose G/xy is 2-connected (the same argument will hold if it is merely 1-connected). Let $\{z_1, z_2\}$ be the set of separating vertices. Now, it must be that one of these vertices resulted from the contraction of the edge xy (otherwise, $\{z_1, z_2\}$ is a separating set of size 2 for G). Without loss of generality, $\{x, y, z_1\}$ is a separating set for G. Hence, $\{z_1\}$ is a separating set for $G - \{x, y\}$, which is a contradiction with the fact that $G - \{x, y\}$ is 2-connected. ☐

3 Problem 3

Proposition 3.1. *Let k be an integer. Any two partitions of a finite set into k-sets admit a common choice of representatives.*

Proof. Let $\mathcal{P}_1$ and $\mathcal{P}_2$ be two partitions of the same finite set into k-sets. Define the graph $G = (V, E)$ where

$$V = \mathcal{P}_1 \cup \mathcal{P}_2$$

$$uv \in E \text{ whenever } u \cap v \neq \emptyset.$$

As $\mathcal{P}_1$ and $\mathcal{P}_2$ are partitions, there can be no edges between vertices corresponding to subsets of the same partition. Hence, G is bipartite with partite sets $\mathcal{P}_1$ and $\mathcal{P}_2$. Now, the question of determining whether or not these two partitions admit a common choice of representatives is equivalent to establishing that G possesses a matching. By Hall's Theorem, we need only verify that G satisfies $|N(S)| \geq |S|$ for all $S \subseteq \mathcal{P}_1$ (the marriage condition). Observe first that for any $S \subseteq \mathcal{P}_1$, if we view the elements of S as subsets of the partition $\mathcal{P}_1$, we have

$$\bigcup_{A \in S} A \leq \bigcup_{A \in N(S)} A$$

by definition of our adjacency relation (if an element of our finite set is contained in some subset in S, then it must be contained in some subset of $N(S)$). Since all subsets are of size k, the equation above ensures that $|N(S)| \geq |S|$, and so the marriage condition is satisfied. Hence, G possesses a 1-factor. Therefore, any two partitions of a finite set into k-sets admit a common choice of representatives. □

4 Problem 4

Definition 4.1. *A graph G is called (vertex-) transitive if, for any two vertices $v, w \in G$, there is an automorphism of G mapping v to w.*

Proposition 4.2. *Every transitive connected graph of even order contains a 1-factor.*

Proof. Let M be a matching in G of maximum size. For any graph, there is $S \subseteq V(G)$ such that

i. S is matchable to $\mathcal{C}_{G-S}$;

ii. Every component of $G - S$ is factor-critical.

From the remarks in Diestel, we know that the maximum matching M is obtained by taking the edges guaranteed by the matchability of S to $\mathcal{C}_{G-S}$ as well as the 1-factor of $C - v$ for each $C \in \mathcal{C}_{G-S}$, where v is the vertex already matched to S.
Suppose now, for the purpose of contradiction, that M does not constitute a 1-factor. By the remarks above, it must be that there is some component of $\mathcal{C}_{G-S}$ that is not matched to S. By the transitivity of G, there is an automorphism mapping a vertex in this component into S. In this isomorphic copy of G, every edge of M is maintained. As before, we are guaranteed that S is matchable to $\mathcal{C}_{G-S}$. Taking M together with this new edge constitutes a larger matching, which is a contradiction with the maximality of M. Therefore, it must be that M is indeed a 1-factor. □

5 Problem 5

Proposition 5.1. *A graph G contains k independent edges if and only if $q(G-S) \leq |S| + |G| - 2k$ for all sets $S \subseteq V(G)$.*

Proof. ($\Rightarrow$) (Partial) Let G contain k independent edges. If we consider the vertex set S given to us by Theorem 2.2.3, we have

$$\begin{aligned} k &\leq |M| \\ &= \frac{1}{2}(|G| + |S| - |\mathcal{C}_{G-S}|) \\ 2k &\leq |G| + |S| - |\mathcal{C}_{G-S}| \\ |\mathcal{C}_{G-S}| &\leq |G| + |S| - 2k. \end{aligned}$$

I do not see how to apply this fact to general subsets of $V(G)$.
($\Leftarrow$) Let G be such that $q(G-S) \leq |S|+|G|-2k$ for all sets $S \subseteq V(G)$. In particular, this inequality holds for the vertex set S_0 satisfying

i. S_0 is matchable to $\mathcal{C}_{G-S_0}$;

ii. Every component of $G - S_0$ is factor-critical,

which is guaranteed to exist. From the remarks in Diestel, we know that the maximum matching M is obtained by taking the edges guaranteed by the matchability of S_0 to $\mathcal{C}_{G-S_0}$ as well as the 1-factor of $C - v$ for each $C \in \mathcal{C}_{G-S_0}$, where v is the vertex already matched to S_0. It follows that

$$\begin{aligned} |M| &= |S_0| + \frac{1}{2}(|G| - |S_0| - |\mathcal{C}_{G-S_0}|) \\ &\geq |S| + \frac{1}{2}(|G| - |S| - (|S| + |G| - 2k)) \\ &= k. \end{aligned}$$

Therefore, G possesses at least k disjoint edges. □

6 Problem 6

Given a graph G, let $\alpha(G)$ denote the largest size of a set of independent vertices in G.

Lemma 6.1. *Let G be a connected, leafless graph. There exists a cycle C of G a vertex $v \in V(C)$ such that $N(v) \subseteq V(C)$.*

Proof. Let $P = v_0v_1 \cdots v_m$ be a longest path in G. Since G is leafless, it must be that v_m has degree at least 2. As P is of maximal length in G, it must be that the neighbors of v_m all lie on P (otherwise, P adjoined with some neighbor results in a path with length strictly greater than that of P). Let v_k be the neighbor of v_m of smallest index and consider the cycle C defined by $v_kPv_mv_k$. By construction, every neighbor of v_m is some v_l with $l \geq k$, and so every neighbor of v_m lies on C, as desired. □

Proposition 6.2. *The vertices of G can be covered by at most $\alpha(G)$ disjoint subgraphs, each isomorphic to a cycle or a K^2 or a K^1.*

Proof. (by strong induction on $\alpha(G)$)
Base Step
If $\alpha(G) = 1$, then G is the complete graph, and so possesses a Hamiltonian cycle (i.e. a single cycle covering all the vertices of G).

Inductive Step
Suppose $\alpha(G) = k$ that the vertices of any graph G' can be covered by at most $\alpha(G')$ disjoint subgraphs whenever $\alpha(G') < k$. We proceed by finding a subgraph H of G ismorphic to one of K_1, K_2, or a cycle containing a vertex v with $N(v) \subseteq V(H)$. If we can produce such a subgraph, observe that $\alpha(G - H) < \alpha(G)$ (the vertex $v \in H$ can be added to any independent set in $G - H$ to make a strictly larger independent set in G). By the inductive hypothesis, $G - H$ can be covered by at most $\alpha(G) - 1$ disjoint copies of K_1, K_2, or cycles, which implies that G can be covered by at most $\alpha(G)$ disjoint copies of these graphs (since H and $G - H$ are disjoint and H is isomorphic to one of the desired graphs).
It remains to show that such a subgraph H indeed exists for any graph.
Suppose G contains an isolated vertex v. We take $H = \{v\}$. As v has no neighbors, it trivially satisfies the condition that $N(v) \subseteq V(H)$.
Supopse G contains a leaf v. Denote its unique neighbor in G by u. We take $V(H) = \{u, v\}$ and $E(H) = \{uv\}$. It follows that $N(v) = \{u\} \subset V(H)$.
If G contains no isolated vertices nor leaves, then we take H to be the cycle guaranteed by 6.1. □

7 Problem 8

Definition 7.1. *A split graph $G = (V, E)$ is a graph whose vertex set admits a partition $V = C \cup I$ so that $G[C]$ is complete and $G[I]$ is an independent set.*

Proposition 7.2. *Any connected split graph with $I \neq \emptyset$ satisfies $\kappa(G) = \lambda(G) = \delta(G)$.*

Proof. Let G be a connected split graph and let $V(G) = C \cup I$ as in the definition. Consider a vertex v of minimum degree in G.
If $v \in C$, then $\deg v = |C|$. Furthermore, each vertex in I has degree equal to $|C|$ (as $\deg v$ is minimum), and so must be adjacent to every vertex in C. Hence, in order to disconnect G, we must remove either the $|C|$ edges incident to a single vertex or we must remove all $|C|$ vertices from C. Thus, in the case where $v \in C$, $\kappa(G) = \lambda(G) = \delta(G)$.
If $v \in I$, then $\deg v \leq |C|$. Furthermore, the component of $G - C$ containing v is of size 1. Hence, v can only be separated from G by removing the $\deg v$ edges incident to v or the $\deg v$ vertices in C that are adjacent to v. Now, as $G[C]$ is complete, we know that $\kappa(G[C]) = \lambda(G[C]) = |C|$, which is at least as big as $\deg v$. Hence, the

edge cutset and vertex cutset described above are in fact minimal. Thus, in the case where $v \in I$, $\kappa(G) = \lambda(G) = \delta(G)$. □

8 Problem 9

Definition 8.1. *The matching number $\nu(G)$ of a graph G is the size of a maximum matching in G.*

Proposition 8.2. *A connected graph G is factor-critical if and only if $\nu(G) = \nu(G - u)$ for every $u \in V(G)$.*

Proof. ($\Rightarrow$) Let G be factor-critical. For any $u \in V(G)$, $G - u$ contains a 1-factor, and so $\nu(G-u) = \frac{1}{2}(|V(G)| - 1)$. As G is of odd degree (all factor-critical graphs are of odd degree), G itself can contain no 1-factor. Hence, the number of vertices in a maximum matching of G can be at most $|V(G)| - 1$, but this is obtained by the matching demonstrated in $G - u$ for any $u \in V(G)$. Hence, $\nu(G) = \frac{1}{2}(|V(G)| - 1)$. Therefore, $\nu(G) = \nu(G - u)$ for any $u \in V(G)$.
($\Leftarrow$) (Partial) Let G be such that $\nu(G) = \nu(G - u)$ for every $u \in V(G)$. Let M be a matching in G of maximum size. For any graph, there is $S \subseteq V(G)$ such that

i. S is matchable to $\mathcal{C}_{G-S}$;

ii. Every component of $G - S$ is factor-critical.

From the remarks in Diestel, we know that the maximum matching M is obtained by taking the edges guaranteed by the matchability of S to $\mathcal{C}_{G-S}$ as well as the 1-factor of $C - v$ for each $C \in \mathcal{C}_{G-S}$, where v is the vertex already matched to S. This gives

$$|M| = |S| + \frac{1}{2}(|G| - |S| - |\mathcal{C}_{G-S}|).$$

Now, let $u \in S$ and consider a maximum matching M' of $G - u$. By hypothesis, it must be that $|M| = |M'|$.
My problem here similar to that in problem 5. I can say something about the particular vertex set S given by Theorem 2.2.3, but I cannot seem to generalize to an arbitrary subset of $V(G)$. □

9 Problem 10

Definition 9.1. *We will say that a graph possesses property T if it is a finite, connected graph in which every block is a triangle and every vertex is of degree 2 or 4.*

Lemma 9.2. *Let G_1 and G_2 be factor critical. The graph obtained by adding an edge between any vertex in G_1 and any vertex in G_2 contains a 1-factor.*

Proof. Let G be constructed as above. Let $u \in G_1$ and $v \in G_2$ be the endvertices of the edge between G_1 and G_2. As G_1 is factor-critical, $G_1 - u$ contains a 1-factor M_1. Similarly, $G_2 - v$ contains a 1-factor M_2. Now, u is unmatched in M_1 and v is unmatched in M_2, so the set $M_1 \cup M_2 \cup \{uv\}$ is a 1-factor of G. $\square$

Proposition 9.3. *If G possesses property T, then G is factor critical.*

Proof. (by strong induction on $|G|$)
Base Step
The smallest graph with property T is K_3. Observe that the removal of any vertex from K_3 leaves a single edge remaining which is a 1-factor its indicent vertices. Hence, K_3 is factor-critical.

Inductive Step
Let G possess property T. To see that G is factor-critical, consider the removal of any vertex v of G.
If v is of degree 2, then its removal does not disconnect the graph (v belongs to a single K_3, which is 2-connected). Furthermore, v belongs to a triangular block, and so its removal leaves only a single edge remaining in the block. Hence, the graph resulting from the removal of v is a pair of graphs G_1 and G_2 connected by a bridge. Now, each of G_1 and G_2 either possesses property T or is a single vertex. If either of G_1 or G_2 is a single vertex, we will relax our defintion slightly to allow them to be called factor-critical. Otherwise, we invoke the inductive hypothesis (as $|G_1|$ and $|G_2|$ are both strictly less than $|G|$) to conclude that they are factor-critical. By Lemma 10.2, $G - v$ contains a 1-factor.
If v is of degree 4, then its removal is equivalent to the removal of two overlapping triangular blocks. Hence, this removal may disconnect the graph into two components. Treating each component separately, we can view each as having subgraphs G_1 and G_2 connected by a bridge with G_1 and G_2 both possessing property T or being a single vertex. By the argument above, each component of $G - v$ contains a 1-factor. Therefore, for any $v \in G$, $G - v$ contains a 1-factor. That is, G is factor-critical. $\square$