Spanning Subgraph Isomorphism

Drew Meier, Dr. Austin Mohr, Thomas Schuler

Department of Mathematics
Nebraska Wesleyan University

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Outline

1. What is Subgraph Isomorphism
2. Cut Vertex Refinement
3. Perfect Matching Refinements
4. Skeleton
5. Data

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- Ullman (1976)

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We use a matrix, C, to hold information about candidate vertices, whose rows correspond to vertices of the pattern, P, and columns correspond the vertices of the host, H.

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Trivial Reductions

- Reduction by Degree

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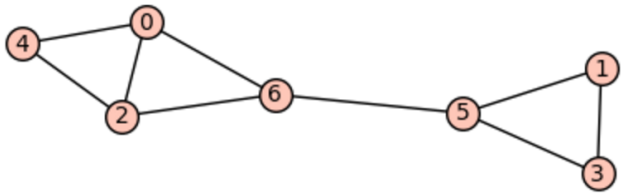
Cut Vertex Refinement

A cut vertex, u, is a cut vertex on a graph, G, if the graph $G - u$ has more components than G.



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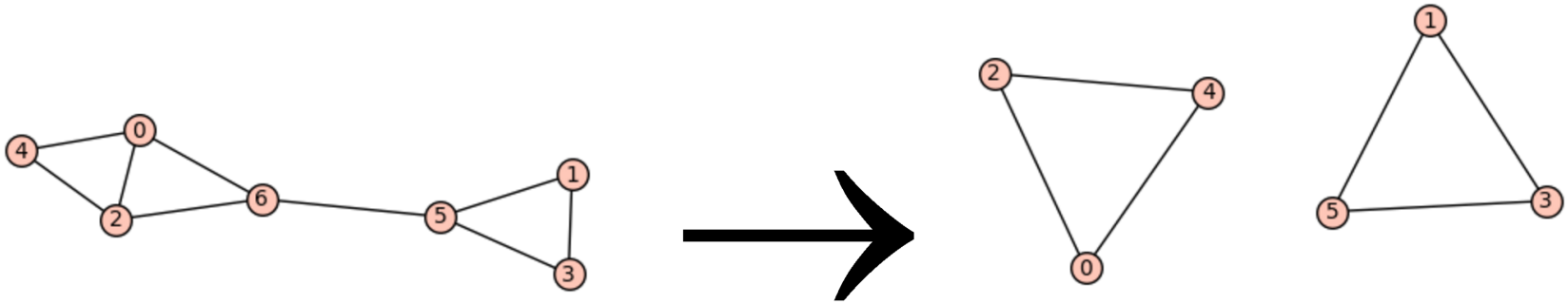

Cut Vertex Refinement

Theorem

A cut vertex, u, of the host, H, must be mapped to a cut vertex, v, of the pattern P

Proof.

Suppose u is a cut vertex of the host graph, H. Now suppose we assign a non-cut vertex v of the pattern, P, to u. Consider only the connected components of H and P containing u and v, H_u and P_v. Since u is a cut vertex, $H_u - u$ has at least 2 components. Since v is not a cut vertex, $P_v - v$ is connected. Clearly, no mapping of the connected pattern, $P_v - v$, into the disconnected host $H_u - u$ can exist. Thus, a cut vertex of the host graph must map to a cut vertex of the pattern graph. □

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Component Partition Refinement

Definition

Let $a = (a_1, a_2, ..., a_n)$ and $b = (b_1, b_2, ..., b_k)$ be two partitions of a common integer. We say the partition a **refines** the partition b provided there is a partition of the indexing set $\{1, ..., n\}$ into blocks $J_1, J_2, ..., J_\ell$ such that $b_i = \sum_{j \in J_i} a_j$ for all $1 \leq i \leq k$

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Thus, a refines b.

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Given a graph G of order n and a vertex u, let $G_1, ..., G_k$ denote the connected components of $G - u$ labeled so that $|G_1| \geq |G_2| \geq \cdots \geq |G_k|$. The **component partition** $C(v)$ is the partition of $n - 1$ given by $(|G_1|, |G_2|, ..., |G_k|)$.

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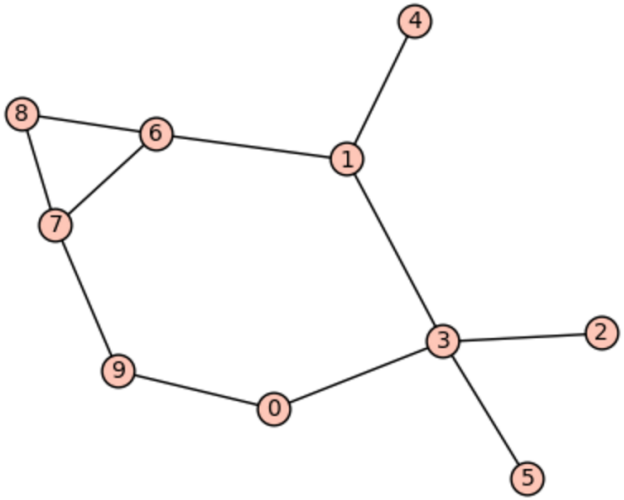
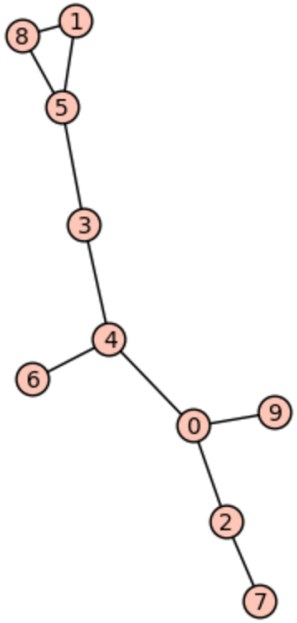
Theorem

Let P and H be graphs of equal order and $f : V(P) \to V(H)$ be a subgraph isomorphism. For any vertex $u \in V(H)$, the component partition $C(f^{-1}(v))$ refines the component partition $C(u)$.

Component Partition Refinement (Example)





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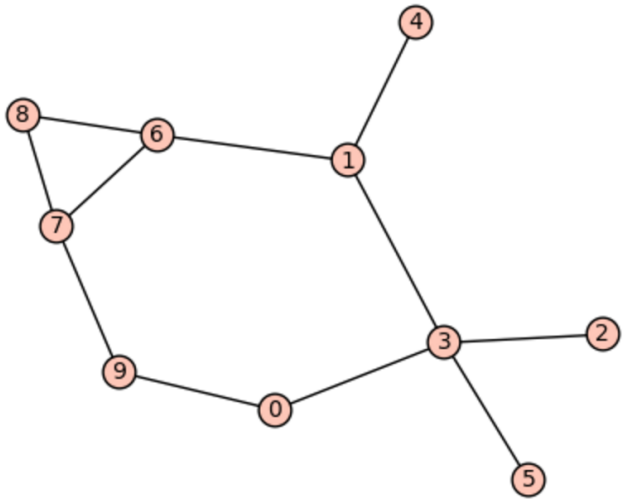

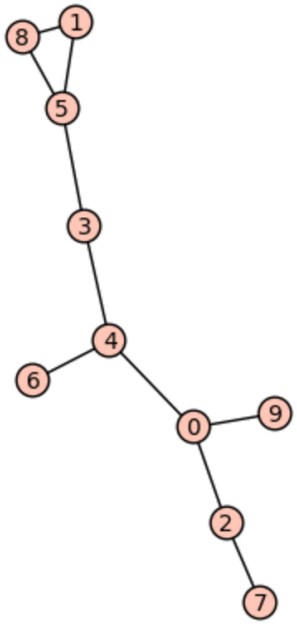

$$\begin{pmatrix} 0 & 1 & 0 & 1 & 0 & 0 & 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 1 & 0 & 0 & 1 & 1 & 1 & 1 \\ 1 & 1 & 0 & 1 & 0 & 0 & 1 & 1 & 1 & 1 \\ 1 & 1 & 0 & 1 & 0 & 0 & 1 & 1 & 1 & 1 \\ 0 & 1 & 0 & 1 & 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 & 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 0 & 1 & 0 & 0 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \end{pmatrix}$$

Degree Reduction

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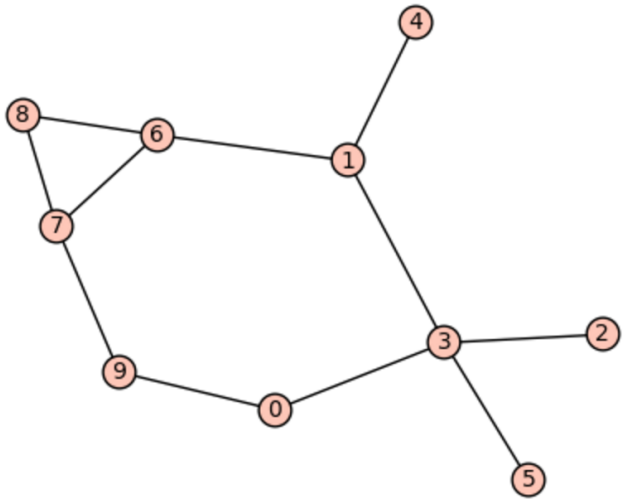

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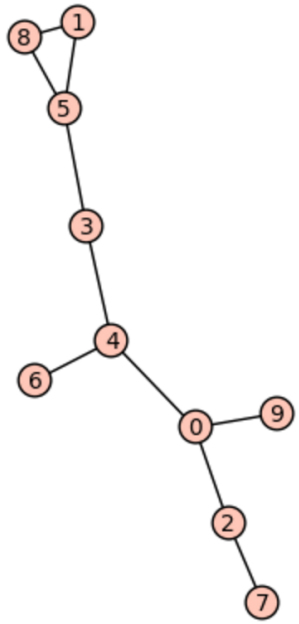

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Cut Vertex Reduction

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A *perfect matching* on a graph, G, is a matching on G such that all vertices of G are visited by an edge in the set.

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Theorem

For vertex u to be a candidate for vertex v upon assignment of v to u, a matching on $K_{u,v}$ the size of the partition v must exist.

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We set up a bipartite graph with the same procedure as with the matching reduction; however, the partitions are the entire set of vertices for H and P. We call this graph, $K_{H,P}$

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Given $K_{H,P}$, for there to exist a subgraph isomorphism from P to H, a perfect matching must exist on $K_{H,P}$.

Proof.

By way of contradiction suppose there is no perfect matching on $K_{H,P}$. Since the partitions of $K_{H,P}$ are of equal order, this implies at least one vertex of each partition has no edge connecting them. This means at least one vertex from P cannot be mapped to any vertex from H. Since at least one vertex from the pattern is left out regardless of which edge configuration we choose for the matching, at least one vertex cannot be mapped. □

Perfect Matching Requirement (Example)

$$\begin{pmatrix} 0 & 0 & 1 & 0 & 1 \\ 1 & 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & 0 & 1 \\ 1 & 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & 0 & 1 \end{pmatrix}$$

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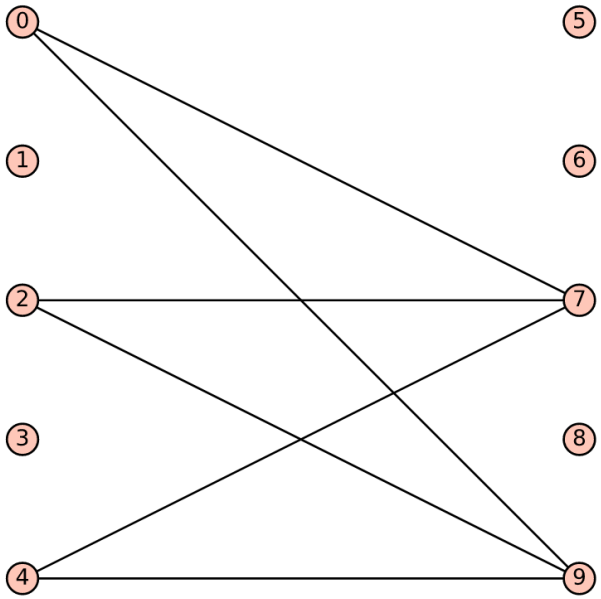

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- Repeat until search tree is exhausted

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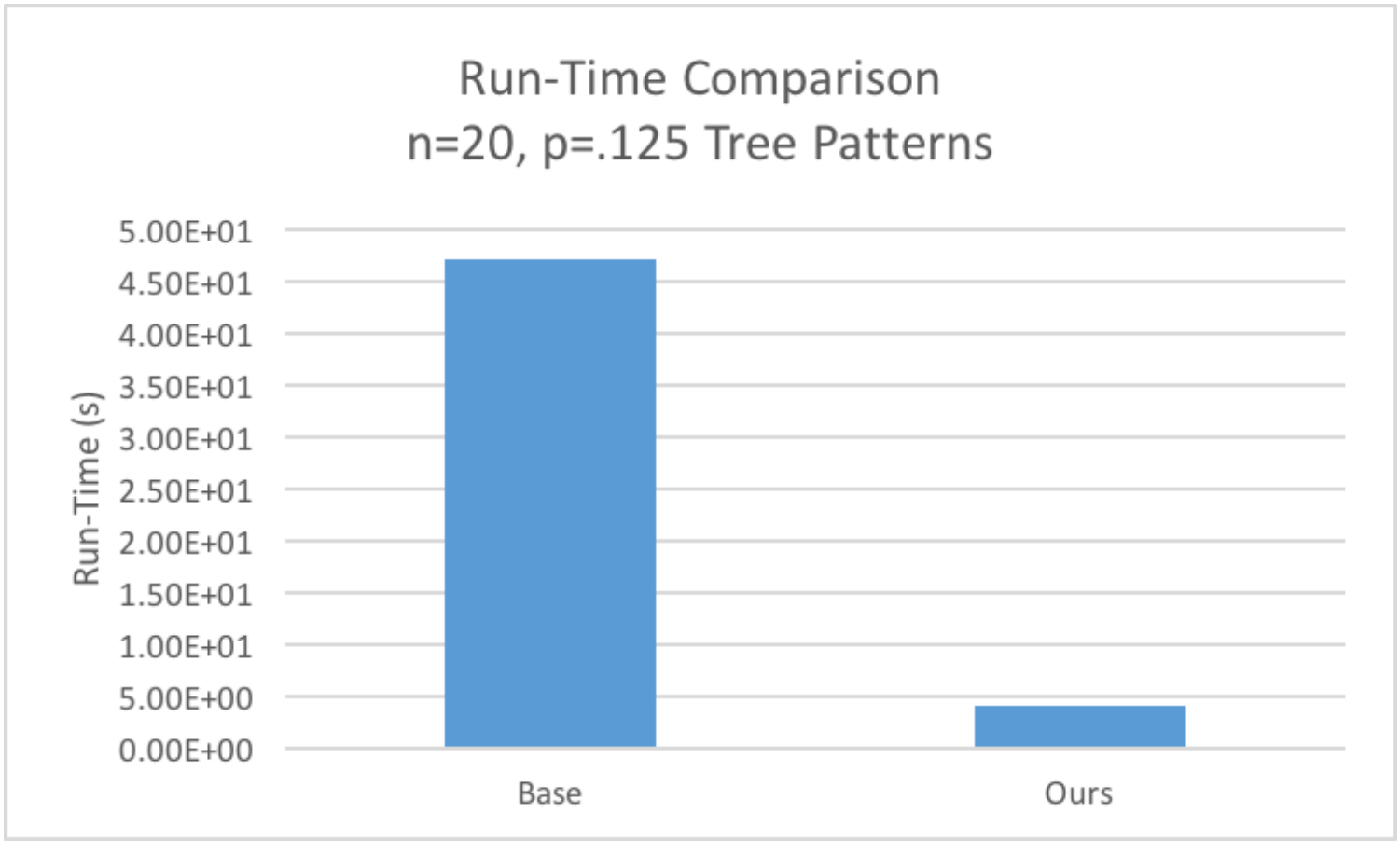
Run-Time Comparison
n=20, p=.125 Tree Patterns
Run-Time (s)
5.00E+01
4.50E+01
4.00E+01
3.50E+01
3.00E+01
2.50E+01
2.00E+01
1.50E+01
1.00E+01
5.00E+00
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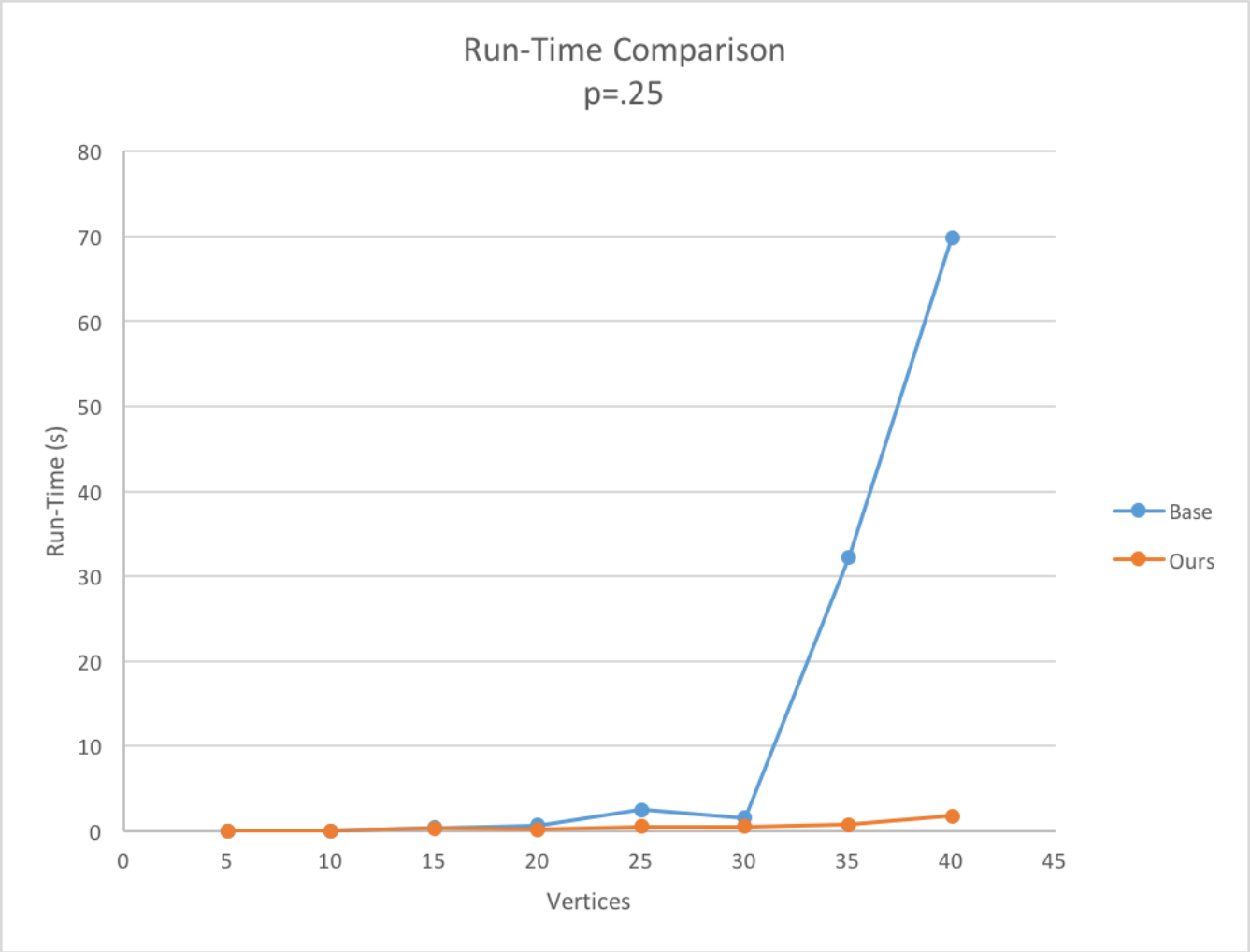
Base
Ours

Data


Run-Time Comparison
p=.25
Run-Time (s)
80
70
60
50
40
30
20
10
0
0
5
10
15
20
25
30
35
40
45
Vertices
Base
Ours

References

- Douglas West, *Introduction to Graph Theory*, Prentice-Hall, 2nd edition, 2001
- Julian R. Ullman, *An algorithm for subgraph isomorphism*, J. of the ACM **23** (1976), 31-42.

Thanks

Thanks for listening!