Quest for Negative Dependency Graphs*

Linyuan Lu, Austin Mohr, and László Székely

Dedicated to our friend, Professor Konstantin Oskolkov, to mark this cairn of his journey searching beauty and boleti.

Abstract The Lovász Local Lemma is a well-known probabilistic technique commonly used to prove the existence of rare combinatorial objects. We explore the lopsided (or negative dependency graph) version of the lemma, which, while more general, appears infrequently in literature due to the lack of settings in which the additional generality has thus far been needed. We present a general framework (matchings in hypergraphs) from which many such settings arise naturally. We also prove a seemingly new generalization of Cayley's formula, which helps to define negative dependency graphs for extensions of forests into spanning trees. We formulate open problems regarding partitions and doubly stochastic matrices that are likely amenable to the use of the lopsided local lemma.

1 Introduction

The Lovász Local Lemma (hereinafter, "L3") says there is a positive probability that none of the events in some collection occurs, provided the dependencies among them are not too strong. First we state L3 [7, 22, 23, 3]. Given events $A_1,\dots,A_n$ in a probability space, define a *dependency graph* to be a simple graph G on $V(G)=[n]$ (i.e. the set of the names of the events, $\{1,2,\dots,n\}$) satisfying that every event A_i is independent of the elements of the event algebra generated by $\{A_j : ij \notin E(G)\}$.

Lemma 1. (Lovász Local Lemma) [3] *Let $A_1,\dots,A_n$ be events with dependency graph G. If there are numbers $x_1,\dots,x_n \in [0,1)$ such that for all i*

$$\Pr(A_i) \le x_i \prod_{ij\in E(G)} (1-x_j).$$

University of South Carolina at Columbia, Department of Mathematics, Columbia, SC 29208, USA
e-mail: lu@math.sc.edu, mohrat@email.sc.edu, szekely@math.sc.edu.

* This research was supported in part by the NSF DMS contract 1000475. We thank Eva Czabarka for her useful suggestions to the manuscript.

Then

$$\Pr\left(\bigwedge_{i=1}^{n}\overline{A_i}\right) \geq \prod_{i=1}^{n}(1-x_i) > 0.$$

The original lemma of Lovász in [7], which is a special case of Lemma 1 above, was used to prove the 2-colorability of a certain hypergraph, and in general, together with the extension L3 [22], to prove the *existence* of combinatorial objects, for which construction seemed hopeless. Surprisingly, there are recent *constructive* versions of the lemma (e.g., in [19]) that will produce the desired object explicitly for many coloration problems. Define $C_G(i) = \{j \in [n] \mid ij \notin E(G)\}$. Inspection of the—not too difficult—proof of L3 concludes that we use independence in the form of

$$\forall i \in [n]\ \forall S \subseteq C_G(i) \quad \Pr\left(\bigwedge_{j\in S}\overline{A_j}\right) \neq 0 \to \Pr\left(A_i \,\middle|\, \bigwedge_{j\in S}\overline{A_j}\right) = \Pr(A_i)\,. \tag{1}$$

Further analyzing the proof, it turns out, that the inequality below is sufficient, instead of equality in (1):

$$\forall i \in [n]\ \forall S \subseteq C_G(i) \quad \Pr\left(\bigwedge_{j\in S}\overline{A_j}\right) \neq 0 \to \Pr\left(A_i \,\middle|\, \bigwedge_{j\in S}\overline{A_j}\right) \leq \Pr(A_i)\,. \tag{2}$$

Formally, a *negative dependency graph* for given events $A_1,\dots,A_n$ in a probability space is a simple graph G on $V(G) = [n]$ (i.e. the set of the names of the events, $\{1,2,\dots,n\}$) satisfying (2). Observe that every negative dependency graph is a dependency graph, but not vice versa.

Lemma 2. (Lopsided Lovász Local Lemma, L4) *Lemma 1 holds with "dependency graph" relaxed to "negative dependency graph".*

The inequality in (2) explains the term "lopsided" in the literature for L4. Negative dependency graphs and the lopsided version of the lemma were first introduced in [8] by Erdős and Spencer to prove the existence of a certain latin transversal, and independently by Albert, Frieze, Reed [2], following the original Lovász version [7]. L4 as stated above is due to Ku [14]. These papers did not investigate classes of problems where L4 could be applied. Few results are present in the literature making use of a negative dependency graph that is not also a dependency graph.

It is worth mentioning here the following useful equivalents of (2):

$$\forall i \in [n]\ \forall S \subseteq C_G(i) \quad \Pr(A_i) \neq 0 \to \Pr\left(\bigwedge_{j\in S}\overline{A_j} \,\middle|\, A_i\right) \leq \Pr\left(\bigwedge_{j\in S}\overline{A_j}\right), \tag{3}$$

or

$$\forall i \in [n]\ \forall S \subseteq C_G(i) \quad \Pr\left(\bigwedge_{j\in S}\overline{A_j}\right) \neq 0 \to \Pr\left(\overline{A_i}\right) \leq \Pr\left(\bigwedge_{j\in S\cup\{i\}}\overline{A_j} \,\middle|\, \bigwedge_{j\in S}\overline{A_j}\right), \tag{4}$$

and also the following, which takes the form of a correlation inequality:

$$\forall i \in [n]\ \forall S \subseteq C_G(i) \quad \Pr(A_i)\Pr\left(\bigvee_{j\in S} A_j\right) \le \Pr\left(A_i \wedge \left(\bigvee_{j\in S} A_j\right)\right). \tag{5}$$

We note that even more general versions of the Lemma hold, practically with the same proof. Given events $A_1,\dots,A_n$ in a probability space, define a *dependency digraph* $\overrightarrow{G}$ on $V(G)=[n]$ by requiring that every event A_i is independent of the elements of the event algebra generated by $\{A_j : j \in C_{\overrightarrow{G}}(i)\}$, with $C_{\overrightarrow{G}}(i) = \{j : ij \notin E(\overrightarrow{G})\}$; and a *negative dependency digraph graph* $\overrightarrow{G}$ by (2), using $\overrightarrow{G}$ and $C_{\overrightarrow{G}}(i)$ instead of G and $C_G(i)$ in (2).

Lemma 3. (Lopsided Lovász Local Lemma, digraph version $\overrightarrow{\mathbf{L4}}$ [3]) *Lemma 1 holds with "dependency graph" relaxed to "negative dependency digraph".*

2 Examples of Negative Dependency Graphs

2.1 Random Matchings in Complete Uniform Hypergraphs

A *matching* M in a hypergraph is a collection of pairwise vertex disjoint hyperedges. The set of vertices covered by hyperedges of the matching M is denoted by $V(M)$. A matching is *perfect* if every vertex of the underlying hypergraph appears in some hyperedge of the matching. In what follows, we will restrict our attention to matchings of the complete k-uniform hypergraph on N vertices, commonly denoted K_N^k.

A matching of K_N^k is an *r-matching* if it consists of r vertex-disjoint hyperedges. For some fixed integer k, r, N satisfying $k \ge 2$, $r \ge 1$, and $N \ge rk$, let $\Omega_N^{k,r}$ be the uniform probability space over all r-matchings of K_N^k. An r-matching of K_N^k is *maximal* if $r = \lfloor \frac{N}{k} \rfloor$; it is *perfect* if $N = kr$. The space of maximal matchings of K_N^k is simply denoted by Ω_N^k.

Given a matching M in K_N^k, we will be interested in the canonical event A^M containing all r-matchings that extend M. More precisely,

$$A^M = A^M_{N,k,r} = \{M' \in \Omega_N^{k,r} \mid M \subseteq M'\}.$$

We will say that two matchings *conflict*, if they have edges that are neither identical nor disjoint, and that two *canonical events conflict*, whenever the matchings used to define them conflict.

Given a collection $\mathscr{M}$ of matchings (or other combinatorial objects, as we will see later), the *conflict graph* for the collection $\{A^M \mid M \in \mathscr{M}\}$ is the simple graph with vertex set $\mathscr{M}$ and edge set $\{M_1M_2 \mid M_1 \in \mathscr{M}$ and $M_2 \in \mathscr{M}$ are in conflict$\}$. The following theorem was proved in [16] in the special case $k=2$ and N is even:

Theorem 1. *Let $\mathscr{M}$ be a collection of matchings in K_N^k. The conflict graph for the collection of canonical events $\{A^M \mid M \in \mathscr{M}\}$ is a negative dependency graph for the probability space Ω_N^k.*

Theorem 1 can be deduced from the following Lemma, which deals with a more general probability space $\Omega_N^{k,r}$. The proof of the Lemma is postponed.

Lemma 4. *In $\Omega_N^{k,r}$, fix a matching $M \in \mathscr{M}$ with $|M| < r$. Let $\mathscr{J}$ be any collection of matchings from $\mathscr{M}$, whose members do not conflict with M. If $|\cup_{M' \in \mathscr{J}} V(M' \setminus M)| \leq rk - |V(M)| + k - 1$, and $\Pr\left(\bigwedge_{M' \in \mathscr{J}} \overline{A^{M'}}\right) > 0$, then*

$$\Pr\left(A^M \,\middle|\, \bigwedge_{M' \in \mathscr{J}} \overline{A^{M'}}\right) \leq \Pr(A_M). \tag{6}$$

Proof of Theorem 1: Write $N = rk + t$. For the space Ω_N^k of maximum size matchings, we have $0 \leq t \leq k - 1$. In this case, we have

$$|\cup_{M' \in \mathscr{J}} V(M' - M)| \leq N - |V(M)| = rk + t - |V(M)| \leq rk - |V(M)| + k - 1.$$

By Lemma 4, the conflict graph for the collection of canonical events $\{A^M \mid M \in \mathscr{M}\}$ is a negative dependency graph for Ω_N^k. $\square$

The following example shows the conflict graph may not be a negative dependency graph of $\Omega_N^{k,r}$ for $r < \lfloor \frac{N}{k} \rfloor$. Take disjoint $M_1, M_2 \in \Omega_N^{k,r}$, so that $M_1 \cup M_2$ is a matching of size $r+1$. This is possible since $r + 1 \leq \lfloor \frac{N}{k} \rfloor$. By our choice, M_1 and M_2 are not adjacent in the conflict graph. However, we have

$$Pr(\overline{A_{M_1}} | A_{M_2}) = 1 > Pr(\overline{A_{M_1}}),$$

contradicting (3). Nevertheless, we can add orientation and some edges to the conflict graph to turn it into a negative dependency *digraph*.

Theorem 2. *For integers $r \geq 1$, $k \geq 2$, and $N \geq (r+1)k$, let $\mathscr{M}$ be a collection of r-matchings in K_N^k. For any $M \in \mathscr{M}$, let S_M be an arbitrary set of $N - (r+1)k + 1$ vertices not in $V(M)$. Let G be a graph on the vertex set $\mathscr{M}$ where MM' is a directed edge of G if M' conflicts with M or M' contains some vertex in S_M. Then G is a negative dependency digraph for the collection of canonical events $\{A^M \mid M \in \mathscr{M}\}$ in the probability space $\Omega_N^{k,r}$.*

Throughout, we assume that the vertex set of K_N^k is $[N]$. We will view the identity map as an injection from $[N]$ into $[N+s]$ for $s \geq 0$, and also from $V(K_N^k)$ to $V(K_{N+s}^k)$ and from $E(K_N^k)$ to $E(K_{N+s}^k)$. To emphasize the difference in the probability space, we use $A_{N,k,r}^M$ to denote the canonical event induced by the matching M in $\Omega_N^{k,r}$, and use $\Pr_N^{k,r}(\bullet)$ to denote the probability in $\Omega_N^{k,r}$. To simplify our notation, we will write $\Pr(\bullet)$ for $\Pr_N^{k,r}(\bullet)$, if the probability space is $\Omega_N^{k,r}$, and spell out the full notation otherwise.

Lemma 5. *For any integers $r \geq 1$, $k \geq 2$, $N \geq rk$, $N_0 = \min\{N-k, rk-1\}$, and any collection $\mathscr{M}$ of matchings in $K^k_{N_0}$, we have*

$$\Pr{}^{k,r-1}_{N-k}\left(\bigwedge_{M\in\mathscr{M}} \overline{A^M_{N-k,k,r-1}}\right) \leq \Pr{}^{k,r}_{N}\left(\bigwedge_{M\in\mathscr{M}} \overline{A^M_{N,k,r}}\right). \tag{7}$$

Proof. Let $t = N - rk$. We have $0 \leq t \leq N-k$. Given an r-matching $M'' \in \Omega^{k,r}_N$, M'' can be viewed as a hypergraph with r pairwise disjoint hyperedges and t isolated vertices. The *right* end of a hyperedge H is $\max_{i\in H} i$. A hyperedge of M'' is called *rightmost* hyperedge, if it has the largest right end among all hyperedges in M''. An k-edge R can be the rightmost hyperedge of some matching $M'' \in \Omega^{k,r}_N$ if and only if the right end vertex of R is at least rk. For $i = 0,1,2,\ldots,t$, let $\mathscr{R}_i$ be the family of k-edges whose right end is $N-i$. Let $\mathscr{R} = \cup^t_{i=0}\mathscr{R}_i$. Consider the mapping $\psi\colon \Omega^{k,r}_N \to \mathscr{R}$, which maps M'' to its rightmost hyperedge. Clearly, $\cup_{R\in\mathscr{R}}\psi^{-1}(R)$ forms a partition of $\Omega^{k,r}_N$.

Fix an i ($0 \leq i \leq t$) and a k-edge $R \in \mathscr{R}_i$. Easy calculation shows that

$$a_i := \Pr\left(\psi^{-1}(R)\right) = \frac{k!r(t)_i}{(N)_i(N-i)_k}.$$

Direct comparison of terms gives

$$a_0 \geq a_1 \geq \cdots \geq a_t. \tag{8}$$

Since $N_0 \leq kr-1$, the hyperedge R above is not in any matching $M' \in \mathscr{M}$. Define $\mathscr{M}'(R) = \{M' \in \mathscr{M} : V(M') \cap R = \emptyset\}$ and observe that

$$\bigwedge_{M'\in\mathscr{M}} \overline{A^{M'}_{N,k,r}} \wedge \psi^{-1}(R) = \bigwedge_{M'\in\mathscr{M}'(R)} \overline{A^{M'}_{N,k,r}} \wedge \psi^{-1}(R).$$

Let $F = \{N-k+1, N-k+2, \ldots, N\}$ and σ be any permutation of $[N]$ that maps $R \setminus F$ to $F \setminus R$, maps $F \setminus R$ to $R \setminus F$, and leaves other vertices as fixed points. The permutation σ maps $\bigwedge_{M'\in\mathscr{M}'(R)} \overline{A^{M'}_{N,k,r}} \wedge \psi^{-1}(R)$ to $\bigwedge_{M'\in\mathscr{M}'(R)} \overline{A^{M'}_{N,k,r}} \wedge A^F_{N,k,r}$. We have

$$\begin{aligned}
\Pr\left(\bigwedge_{M'\in\mathscr{M}} \overline{A^{M'}_{N,k,r}}\right) &= \sum_{i=0}^{t}\sum_{R\in\mathscr{R}_i} \Pr\left(\bigwedge_{M'\in\mathscr{M}} \overline{A^{M'}_{N,k,r}} \wedge \psi^{-1}(R)\right) \\
&= \sum_{i=0}^{t}\sum_{R\in\mathscr{R}_i} \Pr\left(\bigwedge_{M'\in\mathscr{M}'(R)} \overline{A^{M'}_{N,k,r}} \wedge \psi^{-1}(R)\right) \\
&= \sum_{i=0}^{t}\sum_{R\in\mathscr{R}_i} \Pr\left(\bigwedge_{M'\in\mathscr{M}'(R)} \overline{A^{M'}_{N,k,r}} \wedge A^F_{N,k,r}\right)
\end{aligned}$$

$$\geq \sum_{i=0}^{t} \sum_{R \in \mathscr{R}_i} \Pr\left(\bigwedge_{M' \in \mathscr{M}} \overline{A_{N,k,r}^{M'}} \wedge A_{N,k,r}^{F} \right)$$
$$= \sum_{i=0}^{t} \sum_{R \in \mathscr{R}_i} \Pr\left(\bigwedge_{M' \in \mathscr{M}} \overline{A_{N,k,r}^{M'}} \middle| A_{N,k,r}^{F} \right) \Pr\left(A_{N,k,r}^{F}\right)$$
$$= \Pr_{N-k}^{k,r-1}\left(\bigwedge_{M' \in \mathscr{M}} \overline{A_{N-k,k,r-1}^{M'}} \right) \sum_{i=0}^{t} \sum_{R \in \mathscr{R}_i} \Pr\left(A_{N,k,r}^{F}\right).$$

In the last step, we use the fact that

$$\Pr_N^{k,r}\left(\bigwedge_{M' \in \mathscr{M}} \overline{A_{N,k,r}^{M'}} \middle| A_{N,k,r}^{F} \right) = \Pr_{N-k}^{k,r-1}\left(\bigwedge_{M' \in \mathscr{M}} \overline{A_{N-k,k,r-1}^{M'}} \right).$$

Note that $\Pr\left(A_{N,k,r}^{F}\right) = a_0 \geq a_i = \Pr\left(\psi^{-1}(R)\right)$. We have

$$\sum_{i=0}^{t} \sum_{R \in \mathscr{R}_i} \Pr\left(A_{N,k,r}^{F}\right) \geq \sum_{i=0}^{t} \sum_{R \in \mathscr{R}_i} \Pr\left(\psi^{-1}(R)\right) = \Pr\left(\Omega_N^{k,r}\right) = 1.$$

Thus,

$$\Pr\left(\bigwedge_{M' \in \mathscr{M}} \overline{A_{N,k,r}^{M'}} \right) \geq \Pr_{N-k}^{k,r-1}\left(\bigwedge_{M' \in \mathscr{M}} \overline{A_{N-k,k,r-1}^{M'}} \right).$$

□

Proof of Lemma 4: Fix a matching $M \in \mathscr{M}$ and let $\mathscr{J}$ be any collection of matchings from $\mathscr{M}$ that do not conflict with M. Our aim is to show

$$\Pr\left(A^M \middle| \bigwedge_{M' \in \mathscr{J}} \overline{A^{M'}} \right) \leq \Pr\left(A^M\right).$$

Observe that the inequality holds trivially when $\Pr\left(A^M\right) = 0$. Otherwise, the above formula is equivalent with the following (that is essentially (3))

$$\Pr\left(\bigwedge_{M' \in \mathscr{J}} \overline{A^{M'}} \middle| A^M \right) \leq \Pr\left(\bigwedge_{M' \in \mathscr{J}} \overline{A^{M'}} \right).$$

Let $\mathscr{J}^M = \{M' \setminus M \mid M' \in \mathscr{J}\}$. If $M \in \mathscr{J}$, then the lefthand side of the estimate above is zero, and so we have nothing to do. Assume instead that $M \notin \mathscr{J}$. Since every matching M' in $\mathscr{J}$ is not in conflict with M, the vertex set of $M' \setminus M$ is nonempty and is disjoint from the vertex set of M. Let T be the set of vertices covered by the matching M and U be the set of vertices covered by at least one matching $F \in \mathscr{J}^M$. We have $T \cap U = \emptyset$. Let π be a permutation of $[N]$ mapping T to $\{N-|T|+1, N-|T|+2, \ldots, N\}$. We have $\pi(T) \cap \pi(U) = \emptyset$. Thus, $\pi(U) \subseteq [N-|T|]$. Define $\pi(\mathscr{J}^M)$ to be the collection $\{\pi(F) \mid F \in \mathscr{J}^M\}$. We obtain

$$
\begin{aligned}
\Pr\left(\bigwedge_{M' \in \mathscr{J}} \overline{A^{M'}} \,\middle|\, A^M\right) &= \frac{\Pr\left(\bigwedge_{M' \in \mathscr{J}} \overline{A^{M'}} \wedge A^M\right)}{\Pr\left(A^M\right)} \\
&= \frac{\Pr\left(\bigwedge_{M' \in \mathscr{J}} \overline{A^{M' \setminus M}} \wedge A^M\right)}{\Pr\left(A^M\right)} \\
&= \frac{\Pr\left(\bigwedge_{F \in \mathscr{J}^M} \overline{A^F} \wedge A^M\right)}{\Pr\left(A^M\right)} \\
&= \Pr\left(\bigwedge_{F \in \mathscr{J}^M} \overline{A^F} \,\middle|\, A^M\right) \\
&= \Pr\left(\bigwedge_{\pi(F) \in \pi(\mathscr{J}^M)} \overline{A^{\pi(F)}_{N,k,r}} \,\middle|\, A^{\pi(M)}\right) \\
&= \Pr^{k,r-j}_{N-jk}\left(\bigwedge_{\pi(F) \in \pi(\mathscr{J}^M)} \overline{A^{\pi(F)}_{N-jk,k,r-j}}\right) \quad \text{(with } j = |M| < r\text{)} \\
&\le \Pr\left(\bigwedge_{\pi(F) \in \pi(\mathscr{J}^M)} \overline{A^{\pi(F)}_{N,k,r}}\right) \quad \text{(by Lemma 5)} \\
&= \Pr\left(\bigwedge_{F \in \mathscr{J}^M} \overline{A^{F}_{N,k,r}}\right) \\
&= \Pr\left(\bigwedge_{M' \in \mathscr{J}} \overline{A^{M' \setminus M}_{N,k,r}}\right) \\
&\le \Pr\left(\bigwedge_{M' \in \mathscr{J}} \overline{A^{M'}_{N,k,r}}\right).
\end{aligned}
$$

□

2.2 *Random Matchings in Complete Multipartite Graphs*

Theorem 1 shows how a general class of negative dependency graphs can arise from the space of random perfect matchings of K^k_N. A similar result was shown in [17] for the uniform probability space of maximum size matchings of a complete bipartite graph $K_{s,t}$, with the same definition of "conflict" and "canonical event" as above. This can be viewed as the uniform probability space of random injections from an s-element set into an t-element set (for $s \leq t$), providing a plethora of applications.

This generalizes to multipartite matchings as follows. For details see [18]. Let us be given disjoint sets $U_1, \ldots, U_m$ with $|U_1| \leq |U_i|$ for $1 < i$. Call *edges* the sets H, if $H \subseteq \cup_{i=1}^m U_i$ and for all i, $|H \cap U_i| = 1$. A *matching* is a set of disjoint edges. A matching is of maximum size if it covers all elements of U_1. Consider the uniform probability measure on set of all maximum size matchings. Given a matching M, let A^M denote the event of all maximum size matchings that contain all edges of M. We say that two matchings, M_1 and M_2 are in *conflict*, if they contain edges that are neither identical nor disjoint.

Theorem 3. [18] *Let $\mathscr{M}$ be a collection of multipartite matchings on $U_1, \ldots, U_m$. The conflict graph for the collection of canonical events $\{A^M \mid M \in \mathscr{M}\}$ is a negative dependency graph.*

2.3 Spanning Trees in Complete Graphs

The various matching spaces we have mentioned have in common that a partial matching does not conflict with any element of its corresponding canonical event. Indeed, the proof of Theorem 1 relies heavily on this fact by reducing the problem of extending a given partial matching to the problem of finding matchings on the unmatched vertices only.

Consider now the uniform probability space of all spanning trees of K_N. Given a forest F (i.e. a cycle-free subset of the edges of K_N), the canonical event A^F is the collection of all spanning trees of K_N containing F. We say that two forests *conflict* whenever there are a pair of edges, one in the first forest and one in the second, that intersect in exactly one vertex. In other words, two forests F and F' do *not* conflict if, for every connected component $C \subseteq F$ and $C' \subseteq F'$, C and C' are either identical or disjoint.

Theorem 4. *Let $\mathscr{F}$ be a collection of forests in K_N. The conflict graph for the collection of canonical events $\{A^F \mid F \in \mathscr{F}\}$ is a negative dependency graph.*

Notice that the spanning tree setting stands in stark contrast to the matching and partition settings; a forest conflicts with *every* spanning tree in its corresponding canonical event! The proof of Theorem 4 hinges on two lemmata. The first is a direct generalization of Cayley's theorem, while the second is a special case of Theorem 4.

Lemma 6. *Let us be given a forest F in K_N, which has its components $C_1, C_2, ..., C_m$ on $f_1, f_2, ..., f_m$ vertices. Then, the number of spanning trees T in K_N, such that F is contained by T, is*

$$f_1 f_2 \cdots f_m N^{N-2-\sum_i (f_i - 1)}. \tag{9}$$

Proof. Recall Menon's Theorem (Problem 4.1 in [15]): the number of spanning trees in K_N with prescribed degrees $d_1, d_2, ..., d_N$ in vertices $1, 2, ..., N$, is the multinomial coefficient $\binom{N-2}{(d_1-1),...,(d_N-1)}$. Contracting the components of F to single vertices, T contracts to a spanning tree T^* of $K_{N-\sum_i(f_i-1)}$. Let $v_1, ..., v_m$ denote the result of contraction of $C_1, C_2, \ldots, C_m$, and $u_1, u_2, ..., u_{N-\sum_i f_i}$ the vertices from $1, 2, ..., N$, not covered by F. By Menon's theorem, the number of H spanning trees of $K_{N-\sum_i(f_i-1)}$, with degree d_i in v_i and D_j in u_j, is

$$\binom{N-2-\sum_i(f_i-1)}{(d_1-1),\ldots,(d_m-1),(D_1-1),\ldots,(D_{N-\sum_i f_i}-1)}.$$

Note that every H spanning tree of $K_{N-\sum_i(f_i-1)}$, with degree d_i in v_i and D_j in u_j arises *precisely* $\prod_i f_i^{d_i}$ ways as a contraction T^* from some T spanning tree of K_N. Hence the number of spanning trees T containing F is

$$\sum \binom{N-2-\sum_i(f_i-1)}{(d_1-1),\ldots,(d_m-1),(D_1-1),\ldots,(D_{N-\sum_i f_i}-1)} \prod_i f_i^{d_i},$$

where the summation goes over all $d_1, ..., d_m$ and $D_1, ..., D_{N-\sum_i f_i}$ sequences. The multinomial theorem easily evaluates this summation to the required quantity. □

For the next lemma, we say two forests are in *strong conflict*, if they are not vertex disjoint.

Lemma 7. *Let $\mathscr{F}$ be a collection of forests in K_N. The strong conflict graph for the collection of canonical events $\{A^F \mid F \in \mathscr{F}\}$ is a negative dependency graph.*

Proof. To the prove Lemma 7, we prove (5), where A_i is the set of spanning trees containing the forest F_i, and F_i is not in strong conflict with F_j for any $j \in S$. By inclusion-exlusion,

$$\Pr\left(\bigvee_{j\in S} A_j\right) = \sum_{R\subseteq S, |R|\geq 1} \Pr\left(\bigwedge_{j\in R} A_j\right)(-1)^{|R|-1}$$

and

$$\Pr\left(A_i \wedge \left(\bigvee_{j\in S} A_j\right)\right) = \sum_{R\subseteq S, |R|\geq 1} \Pr\left(A_i \wedge \left(\bigwedge_{j\in R} A_j\right)\right)(-1)^{|R|-1}.$$

Observe that the event $A_i \wedge (\wedge_{j\in R} A_j)$ consists of spanning trees that contain the forest F_i and $G_R = \cup_{j\in R} F_j$. The latter graph is either a forest or contains cycle. In the latter case, the corresponding event is impossible. Finally, we claim

$$\Pr(A_i)\Pr\left(\bigwedge_{j\in R} A_j\right) = \Pr\left(A_i \wedge \left(\bigwedge_{j\in R} A_j\right)\right)$$

either by G_R being impossible (and both sides are zero), or by F_i and G_R being vertex disjoint forests, whose union is a forest again, having as components each and every component of F_i and G_R. Lemma 6 finishes the proof. □

Finally, to prove Theorem 4, let A_i denote again the set of spanning trees containing the forest F_i. We are going to prove (5). If F_i has no strong conflict with any F_j ($j \in S$), then Lemma 7 already gives us the desired result. Now suppose F_i does have strong conflict with some F_j ($j \in S$). Define $F'_j = F_j \setminus F_i$ (we mean the difference of the edge sets). Let A'_j be the event corresponding to F'_j. We have

$$\begin{aligned}
\Pr\left(A_i \wedge \left(\bigwedge_{j\in S} \overline{A_j}\right)\right) &= \Pr\left(A_i \wedge \left(\bigwedge_{j\in S} \overline{A'_j}\right)\right) \\
&= \Pr(A_i)\Pr\left(\bigwedge_{j\in S} \overline{A'_j}\right) \qquad \text{(by Lemma 7)} \\
&\leq \Pr(A_i)\Pr\left(\bigwedge_{j\in S} \overline{A_j}\right).
\end{aligned}$$

□

Note that in Lemma 7 we proved a negative dependency graph with equalities everywhere. However, for providing a *dependency graph* in Lemma 7 we would need to prove additional identities—namely, in (5), identity would be needed even if a few A_j's are changed to $\overline{A_j}$. We do not know whether those identities hold. If yes, then it is likely that independence is lurking in the form of independent choice of entries in the Prüfer code or some other sequence encoding of trees.

There is one more interesting comment to make here. Fix any connected graph G, and two of its edges e and f. In the uniform probability space of the spanning trees of G, the correlation inequality

$$\Pr(A^e)\Pr\left(A^f\right) \geq \Pr\left(A^e \wedge A^f\right) \tag{10}$$

holds [25]. This is the opposite of the inequality that we expect for (5)! There is no contradiction, however, as for $G = K_N$ and disjoint edges, (10) holds with identity, and for two edges sharing a single vertex, we have a conflict and we made no claim.

Change the underlying probability space of spanning trees to the uniform probability space of spanning forests of K_N, and let the canonical event associated with a forest be the set of all spanning forests containing it. Then neither *conflict* nor *strong conflict* of forests define a negative dependency graph for their canonical events—this is in line of the conjecture of Kahn [13] that in every connected graph (10) holds, where A^e is the set of spanning spanning forests containing edge e.

2.4 Upper Ideals in Distributive Lattices

Let X be an N-element set and let Ω_N be the probability space consisting of all subsets of X and equipped with the uniform probability measure. For a fixed subset Y of X, define the *canonical event* A^Y to be the collection of all subsets of X that contain Y. In other words,

$$A^Y = \{Z \in \Omega_N \mid Y \subseteq Z\}.$$

Theorem 5. *Let $\mathscr{Y}$ be a collection of nonempty subsets of an N-element set. The graph with vertex set $\{A^Y \mid Y \in \mathscr{Y}\}$ is an edgeless negative dependency graph for the events A^Y.*

More generally, let Γ be a distributive lattice equipped with the uniform probability measure. For $Y \subseteq \Gamma$, let

$$A^Y = \{Z \in \Gamma \mid Y \leq Z\}.$$

Theorem 6. *Let $\mathscr{Y}$ be a collection of elements of a distributive lattice Γ. The graph with vertex set $\{A^Y \mid Y \in \mathscr{Y}\}$ is an edgeless negative dependency graph for the events A^Y.*

Proof. Clearly Theorem 6 implies Theorem 5, if applied to the subset lattice. We have to show (5) for every $A_i = A^Y$ $(Y \in \mathscr{Y})$ and every $S \subseteq \Gamma \setminus \{Y\}$. Consider the indicator functions of the sets A^Y and $\vee_{U \in S} A^U$. These are increasing $\Gamma \to \mathbb{R}$ functions, to which the FKG inequality [10] applies, providing (5). Note that the FKG inequality follows from the even more general Four Functions Theorem [1, 3]. The special case of (5) for the subset lattice already follows from [21].

2.5 Symmetric Events

We say that the events $A_1, A_2, \ldots, A_n$ are *symmetric*, if the probability of any boolean expression of these events do not change, if we substitute $A_{\pi(i)}$ to the place of A_i simultaneously, for any permutation π of $[n]$. The following theorem was proved in [17]:

Theorem 7. *Assume that the events $A_1, A_2, \ldots, A_n$ are symmetric, and let p_i denote $\Pr(A_1 \wedge A_2 \wedge \ldots \wedge A_i)$ for $i = 1, 2, \ldots, n$ and let $p_0 = 1$. If the sequence p_i is logconvex, i.e. $p_k^2 \leq p_{k-1} p_{k+1}$ for $k = 1, 2, \ldots, n-1$, then these events have an empty negative dependency graph.*

3 Open Problems

3.1 Maximum Size Matchings in Graphs

The concept of canonical event and conflict, as defined in Subsection 2.1 can be extended in the case $k = 2$ for maximum size matchings in any graph G. Theorems 1 (for $k = 2$) and 3 can be interpreted that for the graphs $G = K_n$ and $K_{s,t}$, conflict of canonical events define a negative dependency graph. Not every ambient graph will allow this result [16]. For example, for $G = C_6$, let e and f be any two opposite edges. Notice there are only two perfect matchings in C_6. We have that $\Pr\left(A^{\{e\}}\right) = \frac{1}{2}$, while

$$1 = \Pr\left(A^{\{e\}} \,\middle|\, \overline{A^{\{f\}}}\right) \not\leq \Pr\left(A^{\{e\}}\right).$$

The 3-dimensional hypercube also fails to have this property. However, paths with even number of vertices have this property. Can we possibly classify the graphs that have this property?

3.2 Partition Lattice

The space of perfect matchings of K_N^k can be viewed as the space of partitions of an N-element set in which every block is of size k. Can we still find a negative dependency graph without this restriction on block sizes? To state this question more precisely, we will call a collection of disjoint subsets of an N-element set a *partial partition* and say that two partial partitions *conflict* whenever they have two classes neither disjoint nor identical, i.e. their union is not again a partial partition. (A partial partition may in fact fully partition the underlying set.) The ambient probability space is the space of all partitions of an N-element set (equipped with the uniform distribution), so that the canonical event A^M for a given partial partition M is the collection of all partitions extending M.

Conjecture 1 *Let $\mathscr{M}$ be a collection of partial partitions of an N-element. The conflict graph for the collection of canonical events $\{A^M \mid M \in \mathscr{M}\}$ is a negative dependency graph.*

Despite its apparent similarity to Theorem 1, the proof we gave cannot be applied when there are no restrictions on the block sizes. In particular, the necessary adaptation of Lemma 5, namely

$$\Pr_N\Big(\bigwedge_{M\in\mathscr{M}} \overline{A_N^M}\Big) \leq \Pr_{N+1}\Big(\bigwedge_{M\in\mathscr{M}} \overline{A_{N+1}^M}\Big),$$

may fail in some instances. For example, let $M_1 = \{\{1\},\{2\}\}$, $M_2 = \{\{1,\},\{3\}\}$, and $M_3 = \{\{2\},\{3\}\}$. One can compute by hand that

$$\Pr_3\left(\overline{A_3^{M_1}} \wedge \overline{A_3^{M_2}} \wedge \overline{A_3^{M_3}}\right) = \frac{4}{5}, \quad \text{while} \quad \Pr_4\left(\overline{A_4^{M_1}} \wedge \overline{A_4^{M_2}} \wedge \overline{A_4^{M_3}}\right) = \frac{11}{15}.$$

Theorem 6 is not going to help as the partition lattice is not distributive.

Let $M, M_1, \ldots, M_k$ be partial partitions of an N-element set such that M conflicts with none of the M_i (but M_i may conflict with M_j for $i \neq j$). The required $\Pr\left(A^M \,\middle|\, \bigwedge_{i=1}^k \overline{A^{M_i}}\right) \leq \Pr\left(A^M\right)$ is equivalent to the inequality $\Pr\left(A^M\right) \Pr\left(\bigvee_{i=1}^k A^{M_i}\right) \leq \Pr\left(A^M \wedge (\bigvee_{i=1}^k A^{M_i})\right)$ (see (5)).

Let B_j denote the j^{th} Bell number, which counts the number of partitions of a j-element set. The last inequality can be rewritten as

$$\left|A^M\right| \left|\bigcup_{i=1}^{k} A^{M_i}\right| \leq B_N \left|A^M \cap \left(\bigcup_{i=1}^{k} A^{M_i}\right)\right|. \tag{11}$$

$|A_M| = B_{N-|\cup M|}$, and the other two terms in (11) can be expressed by Bell numbers using inclusion-exclusion, however the expression will depend heavily on which blocks the partial partitions have in common and whether they conflict. If we assume that all the partial partitions M_i have disjoint underlying sets and each cover exactly m elements, (11) turns into

$$B_{N-m} \sum_{i=1}^{k} (-1)^{i+1} \binom{k}{i} B_{N-i\cdot m} \leq B_N \sum_{i=1}^{k} (-1)^{i+1} \binom{k}{i} B_{N-(i+1)\cdot m}. \tag{12}$$

The inequality above has been verified asymptotically in N for small fixed values of m and k with Maple, using the modification of the Moser-Wyman formula for the Bell numbers found in [5]. [5] says that uniformly for $h = O(\ln n)$, as $n \to \infty$,

$$B_{n+h} = \frac{(n+h)!}{r^{n+h}} \frac{e^{e^r-1}}{(2\pi B)^{1/2}} \left(1 + \frac{P_0 + hP_1 + h^2 P_2}{e^r} + \frac{Q_0 + hQ_1 + h^2 Q_2 + h^3 Q_3 + h^4 Q_4}{e^{2r}} + O\left(e^{-3r}\right)\right),$$

where $re^r = n$, $B = (r^2 + r)e^r$, P_i and Q_i are known rational functions of r. P_i and Q_i can be found explicitly in [4].

3.3 Permanent of Doubly Stochastic Matrices

Let $A = (a_{i,j})$ be an $n \times n$ doubly stochastic matrix with non-negative entries. For each $1 \leq i \leq n$, let X_i be independent random variables that select the element j from $\{1, \ldots, n\}$ with probability $a_{i,j}$. Define also B_i to be the event that $X_i = X_j$ for some $j \neq i$.

Conjecture 2 *The collection of events $\{B_i \mid 1 \leq i \leq n\}$ are the vertices of an edgeless negative dependency graph.*

This conjecture is relevant because of the continuing interest in lower bounds for the permanent. Computing the permanent is #P-hard [26] and is hard for the entire polynomial-time hierarchy [24]. Schrijver [20] was the first to give an interesting lower bound for the permanent in the form $\mathrm{per}(\tilde{A}) \geq \prod_{i=1}^{n}\prod_{j=1}^{n}(1-a_{i,j})$, where $\tilde{A}$ is the matrix whose $(i,j)^{\text{th}}$ entry is $a_{i,j}(1-a_{i,j})$. Gurvits [12] has the current best lower bound, extending the ideas of [20]:

$$\mathrm{per}(A) \geq \prod_{i=1}^{n}\prod_{j=1}^{n}(1-a_{i,j})^{1-a_{i,j}}. \tag{13}$$

Let us see what L4 gives, provided Conjecture 2 holds. Interpret X_i as selecting an entry j from row i of the doubly stochastic matrix. From this perspective, B_i is the event that, for some row $j \neq i$, the random variables X_i and X_j selected entries belonging to the same column for rows i and j. The product of n entries (one selected from each row) contributes to the permanent precisely when the chosen columns satisfy the event $\bigwedge_{i=1}^{n}\overline{B_i}$. Thus, $\mathrm{per}(A) = \Pr\left(\bigwedge_{i=1}^{n}\overline{B_i}\right) \geq \prod_{i=1}^{n}\left(1-\Pr\left(B_i\right)\right) = \prod_{i=1}^{n}\Pr\left(\overline{B_i}\right)$. Now, $\overline{B_i}$ is the event that, for all $k \neq i$, the value of X_i differs from the value of X_k. Letting $X_i = j$, the probability that $X_k \neq j$ is $1-a_{k,j}$, since the row sum is 1. Summing over j, we have

$$\Pr\left(\overline{B_i}\right) = \sum_{j=1}^{n} a_{i,j}\prod_{\substack{k=1\\k\neq i}}^{n}\left(1-a_{k,j}\right). \tag{14}$$

Finally, L4 would give the lower bound

$$\mathrm{per}(A) \geq \prod_{i=1}^{n}\sum_{j=1}^{n} a_{i,j}\prod_{\substack{k=1\\k\neq i}}^{n}\left(1-a_{k,j}\right). \tag{15}$$

Let us denote by $F(A)$ ($G(A)$) the lower bound for the permanent in (13) (resp. the conjectured lower bound (15)). Gurvits noted [personal communication] that the inequality between the arithmetic and geometric means immediately implies $F(A) \leq G(A)$:

$$\begin{aligned} G(A) &= \prod_{i=1}^{n}\sum_{j=1}^{n} a_{i,j}\prod_{\substack{k=1\\k\neq i}}^{n}\left(1-a_{k,j}\right) \geq \prod_{i=1}^{n}\prod_{j=1}^{n}\prod_{\substack{k=1\\k\neq i}}^{n}\left(1-a_{k,j}\right)^{a_{i,j}} \\ &= \prod_{k=1}^{n}\prod_{j=1}^{n}\prod_{\substack{i=1\\i\neq k}}^{n}\left(1-a_{k,j}\right)^{a_{i,j}} = \prod_{k=1}^{n}\prod_{j=1}^{n}(1-a_{k,j})^{1-a_{k,j}} = F(A). \end{aligned}$$

Gurvits [12] conjectures that among all non-negative doubly stochastic matrices A, $\mathrm{per}(A)/F(A)$ is maximized by the matrix C, in which $c_{i,i} = 1/2$ and for $i \neq j$ $c_{i,j} = 1/(2n-2)$. In this case $\mathrm{per}(C) = (2+o(1))^{-n}$, $F(C) = (\sqrt{2e}+o(1))^{-n}$, $G(C) = (4\sqrt{e}/3+o(1))^{-n}$. Note that $\mathrm{per}(C)/F(C)$ evaluates to $(\sqrt{e/2}+o(1))^{-n}$, while $\mathrm{per}(C)/G(C)$ evaluates to $(2\sqrt{e}/3+o(1))^{n}$. The bound (15) has terms more similar to the permanent than the terms in (13), possibly making it easier to estimate the performance of the approximation.

Another interesting matrix to compare the bounds is $\frac{1}{n}J$, in which every entry is $1/n$. The famous van der Waerden conjecture stated that the permanent of non-negative doubly stochastic matrices is minimized on $\frac{1}{n}J$, with $per(\frac{1}{n}J) = n!/n^n = (1+o(1))e^{-n}$. This conjecture was proved by Friedland [11] with $o(e^{-n})$ error term, and exactly by Falikman [9] and Egorychev [6]. It is easy to see that both $F(\frac{1}{n}J)$ and $G(\frac{1}{n}J)$ evaluate to $(1+o(1))e^{-n}$.

Evidence for the validity of Conjecture 2 is that it holds for $A = \frac{1}{n}J$ or A is a permutation matrix. For any fixed event B_i and any subset $S = \{k_1, k_2, ..., k_s\}$ of the vertices (with $i \notin S$), we have this generalization of (14):

$$\Pr\left(\bigwedge_{j\in S} \overline{B_j}\right) = \sum_{\substack{T\subseteq[n]\\|T|=s}} \sum_{\substack{\pi:S\to T\\ injection}} b_{k_1\pi(k_1)} b_{k_2\pi(k_2)} \cdots b_{k_s\pi(k_s)} \prod_{\ell\notin S}\big(1-\sum_{t\in T} b_{\ell t}\big). \tag{16}$$

Using (16), for $A = \frac{1}{n}J$, the condition (4) boils down to

$$\begin{aligned}\Pr\left(\overline{B_i}\right) &= \left(1-\frac{1}{n}\right)^{n-1} \le \left(1-\frac{1}{n-s}\right)^{n-s-1} \\ &= \frac{(n)_{s+1} n^{-s-1}(1-\frac{s+1}{n})^{n-s-1}}{(n)_s n^{-s}(1-\frac{s}{n})^{n-s}} = \Pr\left(\bigwedge_{j\in S\cup\{i\}} \overline{B_j} \,\middle|\, \bigwedge_{j\in S} \overline{B_j}\right).\end{aligned}$$

Proving this inequality for arbitrary doubly stochastic A has so far eluded us. An alternative proof to Conjecture 2 with $A = \frac{1}{n}J$ is using Theorem 7 with B_i instead of A_i. Theorem 7 was designed for this in [17], although the context was estimating the number of injections.

References

1. R. Ahlswede, D.E. Daykin, An inequality for the weights of two families of sets, their unions and intersections, *Probability Theory and Related Fields* 43 (3)(1978),183–185.
2. M. Albert, A. Frieze, B. Reed, Multicolored Hamiltonian cycles, *Electronic J. Combinatorics* 2 (1995) #R10, pp. 1–13.
3. N. Alon and J.H. Spencer, *The Probabilistic Method*, third edition, (John Wiley and Sons, New York, 2008).
4. E.R. Canfield, bellMoser.pdf, 6 pages manuscript, also see in `http://www.math.sc.edu/~szekely/Aprilattemptformal.pdf`
5. E.R. Canfield and L.H. Harper, A simplified guide to large antichains in the partition lattice, *Congressus Numerantium* 100 (1994), 81–88.
6. G.P. Egorychev, The solution of van der Waerdens problem for permanents, *Adv. Math.* 42 (1981) 299–305.
7. P. Erdős and L. Lovász, Problems and results on 3-chromatic hypergraphs and some related questions, in *Infinite and Finite Sets*, A. Hajnal et. al., Eds., *Colloq. Math. Soc. J. Bolyai* 11, North Holland, Amsterdam, (1975), 609–627.

8. P. Erdős and J.H. Spencer, Lopsided Lovász local lemma and latin transversals, *Discrete Appl. Math.* 30 (1991) 151–154.
9. D. I. Falikman, Proof of the van der Waerdens conjecture on the permanent of a doubly stochastic matrix, *Mat. Zametki* 29 (6) (1981) 931–938, (in Russian).
10. C.M. Fortuin, J. Ginibre, P.N. Kasteleyn, correlation inequalities on some partially ordered sets, *Commun. Math. Phys.* 22 (1971) 89–103.
11. S. Friedland, A lower bound for the permanent of a doubly stochastic matrix, *Ann. Math.* 110 (1979), 167–176.
12. L. Gurvits, Unharnessing the power of Schrijver's permanental inequality, `arXiv:1106.2844v5 [math.CO]` (2011).
13. J. Kahn, A normal law for matchings, *Combinatorica* 20 (3) (2000) 339–391.
14. C.Y. Ku, Lovász local lemma, `https://sites.google.com/site/kuchengyeaw/`
15. L. Lovász, *Combinatorial Problems and Exercises*, North-Holland, 1993.
16. L. Lu and L.A. Székely, A new asymptotic enumeration technique: The Lovász Local Lemma, `arXiv:0905.3983v3 [math.CO]` (2011).
17. L. Lu and L.A. Székely, Using Lovász Local Lemma in the space of random injections, *Electronic J. Comb.* 14 (2007), R63.
18. A. Mohr, On negative dependency graphs in spaces of generalized random matchings, (unpublished manuscript) `http://austinmohr.com/work/negdep.pdf` (2011).
19. R.A. Moser, G. Tardos, A constructive proof of the general Lovász Local Lemma, *Proceedings of the 41st annual ACM Symposium on Theory of Computing* (2009).
20. A. Schrijver, Counting 1-factors in regular bipartite graphs, *J. Comb. Theory, Series B* 72 (1998) 122–135.
21. P.D. Seymour, On incomparable collections of sets, *Mathematika* 20 (1973) 208–209.
22. J. Spencer, Asymptotic lower bounds for Ramsey functions, *Discrete Mathematics*, 20(1977) 69–76.
23. J.H. Spencer, *Ten Lectures on the Probabilistic Method*, CBMS 52, SIAM (1987).
24. S. Toda, PP is as hard as the polynomial-time hierarchy, *SIAM J. Comput.* 20(1991), 865–877.
25. W.T. Tutte, A problem on spanning trees, *Quart. J. Math. Oxford* 25 (2) (1974) 253–255.
26. L. Valiant, The complexity of computing the permanent, *Theretical Computer Science* 8 (2) (1979) 189–201.