The Lopsided Lovász Local Lemma

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Note on Probability Spaces

For this talk, every a probability space Ω is assumed to be uniform and equipped with the counting measure, so that

$$\Pr(A) = \frac{|A|}{|\Omega|}$$

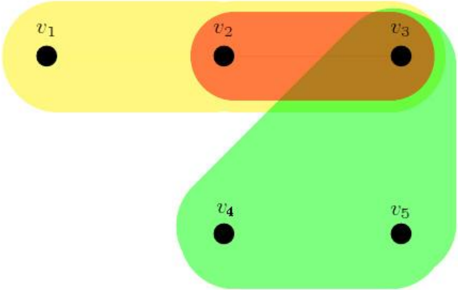
for any subset A of Ω.

Lovász Local Lemma

Given a collection of mostly independent bad events, there is a way to avoid them all.

Hypergraph

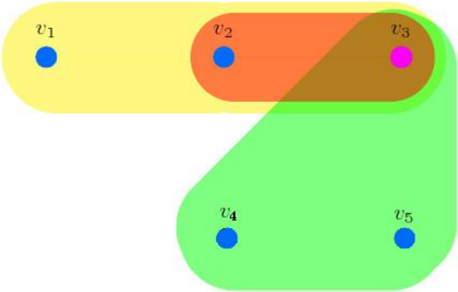
- Ground set V
- Collection E of nonempty subsets of V



$$V = \{v_i \mid i \in [5]\}$$
$$E = \{\{v_1, v_2, v_3\}, \{v_2, v_3\}, \{v_3, v_4, v_5\}\}$$

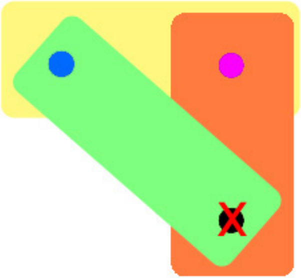
2-Coloring Hypergraphs

A **2-coloring** is an assignment of two colors to the vertices and is **proper** if no edge is monochromatic.



2-Coloring Hypergraphs

If there are many intersections among small edges, a hypergraph may be impossible to 2-color properly.



How many intersections can we allow and still guarantee a proper 2-coloring exists?

2-Coloring Hypergraphs

Color the vertices of a hypergraph H with two colors independently at random.

For each edge e, define the "bad" event

$$A_e = \{\text{2-colorings of } H \mid e \text{ is monochromatic}\}.$$

The proper 2-colorability of H is equivalent to

$$\Pr\left(\bigwedge_{e \in E(H)} \overline{A_e}\right) > 0.$$

2-Coloring Hypergraphs

Let e be an edge of the hypergraph H and F be a collection of edges that are disjoint from e.

The event A_e is independent of the event algebra generated by $\{A_f \mid f \in F\}$.

We capture this information in a **dependency graph**.

2-Coloring Hypergraphs

Dependency Graph G [Erdős, Lovász 1975]

- Each vertex corresponds to an event.
- Each event is independent of the event algebra generated by its non-neighbors in G.

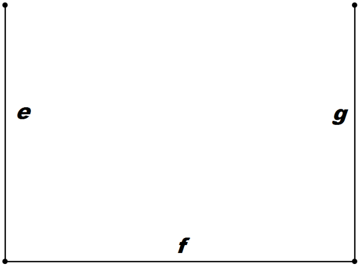
2-Coloring Hypergraphs

For hypergraph 2-coloring, the graph G with

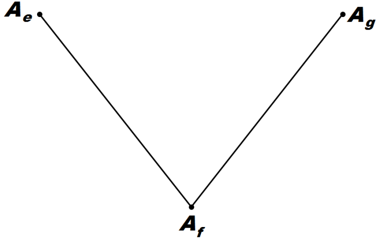
$$V(G) = E(H) \text{ and}$$
$$E(G) = \{ef \mid e \text{ and } f \text{ share a vertex in } H\}$$

is a dependency graph.

Hypergraph to be 2-colored



Dependency Graph



2-Coloring Hypergraphs

Lemma (Symmetric Local Lemma - Erdős, Lovász 1975)

Let $\{A_i \mid i \in [n]\}$ *be a collection of events having a dependency graph* G *such that*

- G *has maximum degree* d *and*
- $\Pr(A_i) \leq p$ *for all* i.

If $ep(d+1) \leq 1$, *then*

$$\Pr\left(\bigwedge_{i=1}^{n} \overline{A_i}\right) > 0.$$

2-Coloring Hypergraphs

For the hypergraph 2-coloring problem,

- $\Pr(A_i) \leq \frac{2}{2^k} = p$ (k is the size of the smallest edge) and
- d is the greatest number of intersections witnessed by any edge.

The local lemma requires

$$ep(d+1) \leq 1$$

and so

$$d \leq \frac{2^{k-1}}{e} - 1.$$

With d bounded, the local lemma concludes

$$\Pr\left(\bigwedge_{i=1}^{n} \overline{A_i}\right) > 0,$$

which means it is possible to properly 2-color the hypergraph.

2-Coloring Hypergraphs

Theorem (Erdős, Lovász 1975)

Let H be a hypergraph in which every edge contains at least k vertices. If each edge intersects at most $\frac{2^{k-1}}{e} - 1$ other edges, then H is properly 2-colorable.

Lovász Local Lemma

Lemma (Asymmetric Local Lemma - Spencer 1977)

Let $\{A_i \mid i \in [n]\}$ be a collection of events having a dependency graph G.
If there are real numbers $x_i \in [0, 1)$ such that

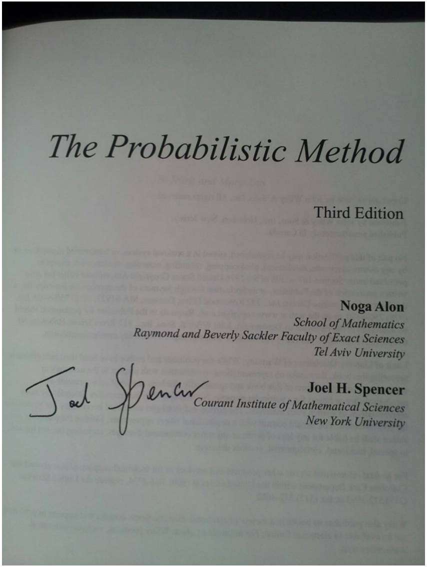
$$\Pr(A_i) \leq x_i \prod_{ij \in E(G)} (1 - x_j)$$

for all i, then

$$\Pr\left(\bigwedge_{i=1}^{n} \overline{A_i}\right) \geq \prod_{i=1}^{n} (1 - x_i) > 0.$$

The symmetric version follows from the asymmetric version by setting each $x_i = \frac{1}{d+1}$.

Jealous?


The Probabilistic Method
Third Edition
Noga Alon
School of Mathematics
Raymond and Beverly Sackler Faculty of Exact Sciences
Tel Aviv University
Joel H. Spencer
Courant Institute of Mathematical Sciences
New York University

Derangements

A **derangement** is a permutation of $[n]$ having no fixed point.

Define the **canonical event**

$$A_i = \{\text{permutations } \pi \text{ of } [n] \mid \pi(i) = i\}.$$

The set $\bigcap_{i=1}^{n} \overline{A_i}$ contains precisely the derangements of $[n]$.

Derangements

The local lemma does not apply, since no pair of the events are independent:

$$\Pr(A_1 \wedge A_2) = \frac{(n-2)!}{n!} = \frac{1}{n^2 - n},$$

while

$$\Pr(A_1)\Pr(A_2) = \frac{(n-1)!}{n!} \cdot \frac{(n-1)!}{n!} = \frac{1}{n^2}.$$

Derangements

Fortunately, derangements possess a different useful property:

$$\Pr(A_1) = \frac{(n-1)!}{n!} = \frac{1}{n}$$

and

$$\Pr\left(A_1 \mid \overline{A_2}\right) = \frac{|A_1 \cap \overline{A_2}|}{|\overline{A_2}|} = \frac{(n-1)! - (n-2)!}{n! - (n-1)!} = \frac{1}{n} - \frac{1}{n(n-1)^2}$$

so

$$\Pr\left(A_1 \mid \overline{A_2}\right) \leq \Pr(A_1)$$

Derangements

In fact,

$$\Pr\left(A_1 \,\middle|\, \bigwedge_{i=2}^{k} \overline{A_i}\right) \leq \Pr\left(A_1\right)$$

for any k.

We capture this information in a **negative dependency graph**.

Derangements

Negative Dependency Graph G [Erdős, Spencer 1991]

- Each vertex corresponds to an event.
- The inequality

$$\Pr\left(A_i \,\middle|\, \bigwedge_{j \in S} \overline{A_j}\right) \leq \Pr(A_i)$$

holds for each event A_i and any subset S of its non-neighbors in G.

Theorem (Lu, Székely 2006)

The graph with vertex set $[n]$ and no edges is a negative dependency graph for the canonical events $\{A_i \mid i \in [n]\}$.

Lopsided Local Lemma

Given a collection of mostly *negative dependent* bad events, there is a way to avoid them all.

Lemma (Lopsided Local Lemma - Erdős, Spencer 1991)

Let $\{A_i \mid i \in [n]\}$ *be a collection of events having negative dependency graph* G*.*
If there are real numbers $x_i \in [0, 1)$ *such that*

$$\Pr(A_i) \leq x_i \prod_{ij \in E(G)} (1 - x_j)$$

for all i*, then*

$$\Pr\left(\bigwedge_{i=1}^{n} \overline{A_i}\right) \geq \prod_{i=1}^{n}(1 - x_i) > 0.$$

Derangements

Setting each $x_i = \frac{1}{n}$, we verify

$$\Pr(A_i) \leq \frac{1}{n} \prod_{ij \in \emptyset} \left(1 - \frac{1}{n}\right)$$

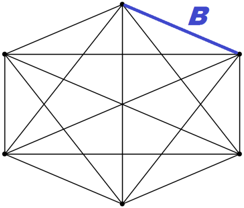
and conclude

$$\Pr\left(\bigwedge_{i=1}^{n} \overline{A_i}\right) \geq \left(1 - \frac{1}{n}\right)^n \xrightarrow{n} \frac{1}{e}.$$

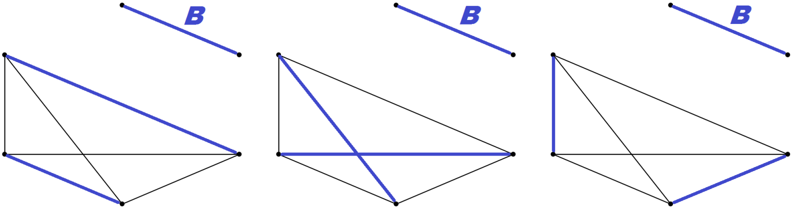
Hypergraph Matchings

The **canonical event** for a partial matching is the collection of all perfect matchings extending it.

partial matching B



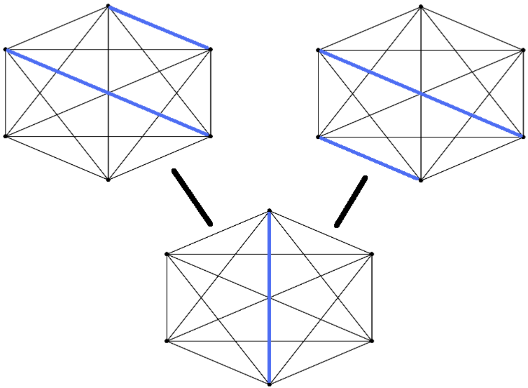
canonical event A_B



Hypergraph Matchings

Conflict Graph

- Each vertex corresponds to a partial matching.
- Two matchings are adjacent if their union is not again a partial matching.



Hypergraph Matchings

Theorem (Lu, M, Székely 2013)

Let $\mathcal{M}$ be any collection of matchings in a complete uniform hypergraph. The conflict graph is a negative dependency graph for the canonical events $\{A_M \mid M \in \mathcal{M}\}$.

The set

$$\bigcap_{M \in \mathcal{M}} \overline{A_M}$$

contains all perfect matchings of the complete uniform hypergraph that extend no matching from $\mathcal{M}$.

Asymptotics from the Lopsided Local Lemma

ϵ-**Near Positive Dependency Graph** G [Lu, Székely 2011]

- Each vertex of G corresponds to an event.
- $\Pr(A_i \wedge A_j) = 0$ whenever $ij \in E(G)$.
- The inequality

$$\Pr\left(A_i \,\middle|\, \bigwedge_{j \in S} \overline{A_j}\right) \geq (1-\epsilon)\Pr(A_i)$$

holds for each event A_i and any subset S of its non-neighbors in G.

Theorem (M 2013)

Let $\mathcal{M}$ be a collection of matchings in a complete uniform hypergraph. If $\mathcal{M}$ is sufficiently "sparse", then the conflict graph for the canonical events $\{A_M \mid M \in \mathcal{M}\}$ is an ϵ-near positive dependency graph.

Asymptotics from the Lopsided Local Lemma

Let $A_1, \ldots, A_n$ be events in a probability space Ω_N that grows with N and set $\mu = \sum_{i=1}^{n} \Pr(A_i)$.

If the probabilities are appropriately controlled, then a negative dependency graph gives the *lower bound*

$$\Pr\left(\bigwedge_{i=1}^{n} \overline{A_i}\right) \geq (1 - o(1))e^{-\mu} \text{ [Lu, Székely 2011]}$$

and a positive dependency graph gives the *upper bound*

$$\Pr\left(\bigwedge_{i=1}^{n} \overline{A_i}\right) \leq (1 + o(1))e^{-\mu} \text{ [M 2013]}$$

as N tends to infinity.

Asymptotics from the Lopsided Local Lemma

Corollary (M 2013)

Let $A_1, \ldots, A_n$ be events in a probability space Ω_N. If the conditions of the previous two theorems are satisfied, then

$$\left|\bigcap_{i=1}^{n} \overline{A_i}\right| = (1 + o(1))e^{-\mu}|\Omega_N|$$

as N tends to infinity.

Counting Hypergraphs by Girth

A typical k-**cycle** in a hypergraph is one in which consecutive edges intersect in exactly one vertex.

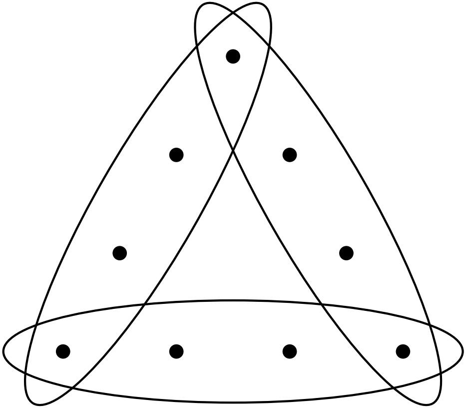
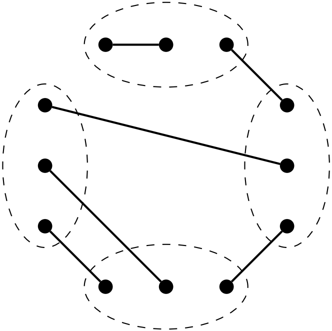


Figure : 3-cycle in 4-uniform hypergraph.

Counting Hypergraphs by Girth

The **configuration model** [Bollobás 1980] connects matchings with multihypergraphs.

2-Matching



Multihypergraph

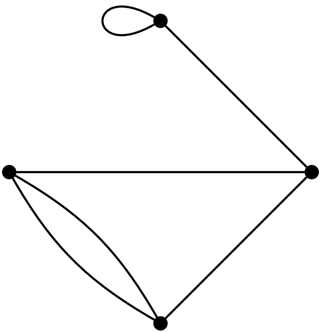


Figure : Configuration projecting to 3-regular, 2-uniform multihypergraph on four vertices.

Counting Hypergraphs by Girth

Let $\mathcal{M}$ contain all matchings whose projection is a k-cycle with $k < g$.

For such a collection, the set

$$\bigcap_{M \in \mathcal{M}} \overline{A_M}$$

contains precisely the matchings that represent hypergraphs of girth at least g in the configuration model.

Counting Hypergraphs by Girth

Theorem (M, Székely 2013+)

The number of r-regular, s-uniform hypergraphs having girth at least g is

$$(1+o(1))\frac{(rN)!}{s!^{rN/s}\left(\frac{rN}{s}\right)!(r!)^N}\exp\left(-\sum_{i=1}^{g-1}\frac{(r-1)^i(s-1)^i}{2i}\right)$$

(assuming g, r, and s grow slowly with N).

Homework

1. Think of your favorite combinatorial object.
   - Partial matchings
2. Define the canonical event for a particlar instance of that type.
   - All perfect matchings extending it
3. Define conflict for two objects of that type.
   - Union is not a matching
4. Determine whether your conflict graph is a negative dependency graph.
   - Ask László

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Thanks

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Further Information

- A. Mohr, *Applications of the Lopsided Lovász Local Lemma Regarding Hypergraphs*, Ph.D. Dissertation (2013).
- L. Lu, A. Mohr, and L. A. Székely, *Quest for Negative Dependency Graphs*, Recent Advances in Harmonic Analysis and Applications (in honor of Konstantin Oskolkov), Springer Proceedings in Mathematics and Statistics **25** (2013).
- L. Lu, A. Mohr, and L. A. Székely, *Connected Balanced Subgraphs in Random Regular Multigraphs Under the Configuration Model*, Journal of Combinatorial Mathematics and Combinatorial Computing (2013+).