The Lopsided Lovász Local Lemma

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For this talk, every a probability space $\Omega$ is assumed to be uniform and equipped with the counting measure, so that

$$\Pr(A) = \frac{|A|}{|\Omega|}$$

for any subset $A$ of $\Omega$. 
Given a collection of mostly independent bad events, there is a way to avoid them all.
2-Coloring Hypergraphs

Hypergraph

- Ground set $V$
- Collection $E$ of nonempty subsets of $V$

$a_1$
$a_2$
$a_3$
$a_4$
$a_5$

$V = \{ a_i \mid i \in [5] \}$
$E = \{ \{a_1, a_2, a_3\}, \{a_2, a_3\}, \{a_3, a_4, a_5\} \}$
A 2-coloring is an assignment of two colors to the vertices and is proper if no edge is monochromatic.

![Graph](image-url)
2-Coloring Hypergraphs

If there are many intersections among small edges, a hypergraph may be impossible to 2-color properly.

How many intersections can we allow and still guarantee a proper 2-coloring exists?
2-Coloring Hypergraphs

Color the vertices of a hypergraph $H$ with two colors independently at random.

For each edge $e$, define the “bad” event

$$A_e = \{2\text{-colorings of } H \mid e \text{ is monochromatic}\}.$$ 

The proper 2-colorability of $H$ is equivalent to

$$\Pr \left( \bigwedge_{e \in E(H)} \overline{A_e} \right) > 0.$$
Let $e$ be an edge of the hypergraph $H$ and $F$ be a collection of edges that are disjoint from $e$.

The event $A_e$ is independent of the event algebra generated by $\{A_f \mid f \in F\}$.

We capture this information in a dependency graph.
2-Coloring Hypergraphs

Dependency Graph $G$ [Erdős, Lovász 1975]

- Each vertex corresponds to an event.
- Each event is independent of the event algebra generated by its non-neighbors in $G$. 
2-Coloring Hypergraphs

For hypergraph 2-coloring, the graph $G$ with

$$V(G) = E(H) \text{ and } E(G) = \{ef \mid e \text{ and } f \text{ share a vertex in } H\}$$

is a dependency graph.
Lemma (Symmetric Local Lemma - Erdős, Lovász 1975)

Let \( \{A_i \mid i \in [n]\} \) be a collection of events having a dependency graph \( G \) such that

- \( G \) has maximum degree \( d \) and
- \( \Pr (A_i) \leq p \) for all \( i \).

If \( ep(d + 1) \leq 1 \), then

\[
\Pr \left( \bigwedge_{i=1}^{n} \overline{A_i} \right) > 0.
\]
For the hypergraph 2-coloring problem,

- \( \Pr (A_i) \leq \frac{2}{2^k} = p \) (\( k \) is the size of the smallest edge) and
- \( d \) is the greatest number of intersections witnessed by any edge.

The local lemma requires

\[
ep(d + 1) \leq 1
\]

and so

\[
d \leq \frac{2^{k-1}}{e} - 1.
\]

With \( d \) bounded, the local lemma concludes

\[
\Pr \left( \bigwedge_{i=1}^{n} \overline{A_i} \right) > 0,
\]

which means it is possible to properly 2-color the hypergraph.
Theorem (Erdős, Lovász 1975)

Let $H$ be a hypergraph in which every edge contains at least $k$ vertices. If each edge intersects at most $2^k - 1 - 1$ other edges, then $H$ is properly 2-colorable.
Lovász Local Lemma

Lemma (Asymmetric Local Lemma - Spencer 1977)

Let \( \{A_i \mid i \in [n]\} \) be a collection of events having a dependency graph \( G \).

If there are real numbers \( x_i \in [0, 1) \) such that

\[
\Pr (A_i) \leq x_i \prod_{ij \in E(G)} (1 - x_j)
\]

for all \( i \), then

\[
\Pr \left( \bigwedge_{i=1}^{n} \overline{A_i} \right) \geq \prod_{i=1}^{n} (1 - x_i) > 0.
\]

The symmetric version follows from the asymmetric version by setting each \( x_i = \frac{1}{d+1} \).
The Probabilistic Method

Third Edition

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A **derangement** is a permutation of \([n]\) having no fixed point.

Define the **canonical event**

\[
A_i = \{ \text{permutations } \pi \text{ of } [n] \mid \pi(i) = i \}.
\]

The set \(\bigcap_{i=1}^{n} \overline{A_i}\) contains precisely the derangements of \([n]\).
The local lemma does not apply, since no pair of the events are independent:

\[ \Pr (A_1 \land A_2) = \frac{(n - 2)!}{n!} = \frac{1}{n^2 - n}, \]

while

\[ \Pr (A_1) \Pr (A_2) = \frac{(n - 1)!}{n!} \cdot \frac{(n - 1)!}{n!} = \frac{1}{n^2}. \]
Fortunately, derangements possess a different useful property:

\[ \Pr(A_1) = \frac{(n-1)!}{n!} = \frac{1}{n} \]

and

\[ \Pr(A_1 \mid \overline{A_2}) = \frac{|A_1 \cap \overline{A_2}|}{|\overline{A_2}|} = \frac{(n-1)! - (n-2)!}{n! - (n-1)!} = \frac{1}{n} - \frac{1}{n(n-1)^2} \]

so

\[ \Pr(A_1 \mid \overline{A_2}) \leq \Pr(A_1) \]
In fact,

\[ \Pr \left( A_1 \left| \bigwedge_{i=2}^{k} \overline{A}_i \right. \right) \leq \Pr (A_1) \]

for any \( k \).

We capture this information in a **negative dependency graph**.
Derangements

**Negative Dependency Graph** $G$ [Erdős, Spencer 1991]

- Each vertex corresponds to an event.
- The inequality

\[
\Pr \left( A_i \left| \bigwedge_{j \in S} \overline{A_j} \right. \right) \leq \Pr (A_i)
\]

holds for each event $A_i$ and any subset $S$ of its non-neighbors in $G$.

**Theorem (Lu, Székely 2006)**

*The graph with vertex set $[n]$ and no edges is a negative dependency graph for the canonical events $\{A_i \mid i \in [n]\}$.*
Given a collection of mostly negative dependent bad events, there is a way to avoid them all.

**Lemma (Lopsided Local Lemma - Erdős, Spencer 1991)**

Let \( \{A_i \mid i \in [n]\} \) be a collection of events having negative dependency graph \( G \).

If there are real numbers \( x_i \in [0, 1) \) such that

\[
\Pr(A_i) \leq x_i \prod_{ij \in E(G)} (1 - x_j)
\]

for all \( i \), then

\[
\Pr \left( \bigwedge_{i=1}^n \overline{A_i} \right) \geq \prod_{i=1}^n (1 - x_i) > 0.
\]
Derangements

Setting each $x_i = \frac{1}{n}$, we verify

$$\Pr(A_i) \leq \frac{1}{n} \prod_{ij \in \emptyset} \left(1 - \frac{1}{n}\right)$$

and conclude

$$\Pr \left( \bigwedge_{i=1}^{n} \overline{A_i} \right) \geq \left(1 - \frac{1}{n}\right)^n \xrightarrow{n \to \infty} \frac{1}{e}.$$
Hypergraph Matchings

The **canonical event** for a partial matching is the collection of all perfect matchings extending it.

![Diagram of a partial matching and a canonical event]
Conflict Graph

- Each vertex corresponds to a partial matching.
- Two matchings are adjacent if their union is not again a partial matching.
Theorem (Lu, M, Székely 2013)

Let $\mathcal{M}$ be any collection of matchings in a complete uniform hypergraph. The conflict graph is a negative dependency graph for the canonical events $\{A_M \mid M \in \mathcal{M}\}$.

The set

$$\bigcap_{M \in \mathcal{M}} \overline{A_M}$$

contains all perfect matchings of the complete uniform hypergraph that extend no matching from $\mathcal{M}$. 

Asymptotics from the Lopsided Local Lemma

$\epsilon$-Near Positive Dependency Graph $G$ [Lu, Székely 2011]

- Each vertex of $G$ corresponds to an event.
- $\Pr(A_i \wedge A_j) = 0$ whenever $ij \in E(G)$.
- The inequality

$$\Pr \left( A_i \bigg\vert \bigwedge_{j \in S} A_j \right) \geq (1 - \epsilon) \Pr(A_i)$$

holds for each event $A_i$ and any subset $S$ of its non-neighbors in $G$.

Theorem (M 2013)

Let $\mathcal{M}$ be a collection of matchings in a complete uniform hypergraph. If $\mathcal{M}$ is sufficiently “sparse”, then the conflict graph for the canonical events $\{A_M \mid M \in \mathcal{M}\}$ is an $\epsilon$-near positive dependency graph.
Asymptotics from the Lopsided Local Lemma

Let $A_1, \ldots, A_n$ be events in a probability space $\Omega_N$ that grows with $N$ and set $\mu = \sum_{i=1}^{n} \Pr(A_i)$.

If the probabilities are appropriately controlled, then a negative dependency graph gives the lower bound

$$\Pr\left(\bigwedge_{i=1}^{n} \overline{A_i}\right) \geq (1 - o(1))e^{-\mu} \quad [\text{Lu, Székely 2011}]$$

and a positive dependency graph gives the upper bound

$$\Pr\left(\bigwedge_{i=1}^{n} \overline{A_i}\right) \leq (1 + o(1))e^{-\mu} \quad [\text{M 2013}]$$

as $N$ tends to infinity.
Corollary (M 2013)

Let $A_1, \ldots, A_n$ be events in a probability space $\Omega_N$. If the conditions of the previous two theorems are satisfied, then

$$\left| \bigcap_{i=1}^{n} \overline{A_i} \right| = (1 + o(1)) e^{-\mu} |\Omega_N|$$

as $N$ tends to infinity.
A typical \textit{k-cycle} in a hypergraph is one in which consecutive edges intersect in exactly one vertex.
The **configuration model** [Bollobás 1980] connects matchings with multihypergraphs.

**Figure**: Configuration projecting to 3-regular, 2-uniform multihypergraph on four vertices.
Let $\mathcal{M}$ contain all matchings whose projection is a $k$-cycle with $k < g$.

For such a collection, the set

$$\bigcap_{M \in \mathcal{M}} \overline{A_M}$$

contains precisely the matchings that represent hypergraphs of girth at least $g$ in the configuration model.
The number of $r$-regular, $s$-uniform hypergraphs having girth at least $g$ is

\[(1 + o(1)) \frac{(rN)!}{s! r^{N/s} \left( \frac{rN}{s} \right)! (r!)^N} \exp \left( - \sum_{i=1}^{g-1} \frac{(r-1)^i (s-1)^i}{2i} \right) \]

(assuming $g$, $r$, and $s$ grow slowly with $N$).
Homework

1. Think of your favorite combinatorial object.
   - Partial matchings

2. Define the canonical event for a particular instance of that type.
   - All perfect matchings extending it

3. Define conflict for two objects of that type.
   - Union is not a matching

4. Determine whether your conflict graph is a negative dependency graph.
   - Ask László
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Further Information

