Diamonds in the Rough

(The Lovász Local Lemma)

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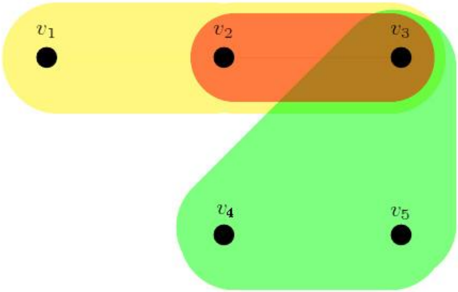
2-Coloring Hypergraphs

Hypergraph

- Ground set V
- Collection E of nonempty subsets of V

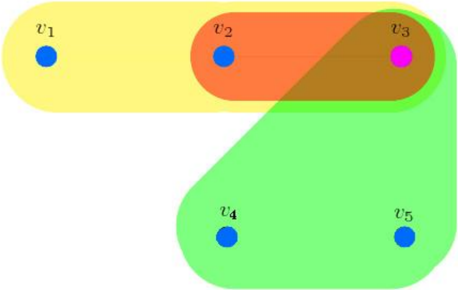


$$V = \{v_i \mid i \in [5]\}$$
$$E = \{\{v_1, v_2, v_3\}, \{v_2, v_3\}, \{v_3, v_4, v_5\}\}$$

2-Coloring Hypergraphs

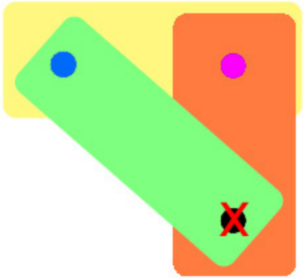
2-Coloring

- Assignment of two colors to vertices
- No edge monochromatic



2-Coloring Hypergraphs

If there are many intersections among the edges, a hypergraph may be impossible to 2-color.



How many intersections can we allow and still guarantee a 2-coloring exists?

2-Coloring Hypergraphs

Color the vertices of a hypergraph H with two colors independently at random.
For each edge e_i, define the event

$$A_i\text{: ``edge } e_i \text{ is monochromatic''}.$$

The 2-colorability of H is equivalent to

$$\Pr\left(\bigwedge_i \overline{A_i}\right) > 0.$$

2-Coloring Hypergraphs

Dependency Graph G

- Each vertex i corresponds to an event A_i.
- Each event A_i is independent of the event algebra generated by its non-neighbors in G.

The graph G whose vertices correspond to the monochromatic events with

$$E(G) = \{ij \mid e_i \sim_H e_j\}$$

is a dependency graph.

2-Coloring Hypergraphs

Lemma (Lovász Local Lemma, Version 1)

Let $\{A_i \mid i \in [n]\}$ *be a collection of events having a dependency graph* G *of maximum degree* d*.*

If

- $\Pr(A_i) \leq p$ *for all* i *and*
- $ep(d+1) \leq 1$,

then

$$\Pr\left(\bigwedge_{i=1}^{n} \overline{A_i}\right) > 0.$$

2-Coloring Hypergraphs

Theorem (Erdős-Lovász)

Let H be a hypergraph in which every edge has size at least k. If each edge intersects at most $(2^{k-1}/e) - 1$ other edges, then H is 2-colorable.

2-Coloring Hypergraphs

Proof.

Set $p = 1/2^{k-1}$ and $d = (2^{k-1}/e) - 1$.

- Dependency graph has maximum degree at most d (since each edge intersects at most $(2^{k-1}/e) - 1$ other edges)
- $\Pr(A_i) \leq p$ for all i (since every edge has size at least k)
- $ep(d+1) = e(1/2^{k-1})(2^{k-1}/e) = 1$

The Lovász Local Lemma gives $\Pr\left(\bigwedge_{i=1}^{n} \overline{A_i}\right) > 0$. That is, H is 2-colorable. □

2-Coloring Hypergraphs

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The Lovász Local Lemma gives $\Pr\left(\bigwedge_{i=1}^{n} \overline{A_i}\right) > 0$. That is, H is 2-colorable. □

Lovász Local Lemma

Lemma (Lovász Local Lemma, Version 2)

Let $\{A_i \mid i \in [n]\}$ *be a collection of events having a dependency graph* G*.*

If there are real numbers $x_i \in [0, 1)$ *such that, for all* i*,*

$$\Pr(A_i) \leq x_i \prod_{ij \in E(G)} (1 - x_j),$$

then

$$\Pr\left(\bigwedge_{i=1}^{n} \overline{A_i}\right) > 0.$$

Proof of Lovász Local Lemma, Version 2

Claim

Let $S \subseteq [n]$*. For any* $i \notin S$*,*

$$\Pr\left(A_i \,\middle|\, \bigwedge_{j \in S} \overline{A_j}\right) \le x_i.$$

Proof of Lovász Local Lemma, Version 2

(Proof of claim by induction on $|S|$.)
Set $S_1 = \{j \in S \mid ij \in E(G)\}$ and $S_2 = S \setminus S_1$. Now,

$$\Pr\left(A_i \;\middle|\; \bigwedge_{j \in S} \overline{A_j}\right) = \frac{\Pr\left(A_i \wedge \bigwedge_{j \in S_1} \overline{A_j} \;\middle|\; \bigwedge_{k \in S_2} \overline{A_k}\right)}{\Pr\left(\bigwedge_{j \in S_1} \overline{A_j} \;\middle|\; \bigwedge_{k \in S_2} \overline{A_k}\right)}.$$

We will bound the numerator and denominator separately.

Proof of Lovász Local Lemma, Version 2

For the numerator, use the fact that A_i is independent of $\{A_k \mid k \in S_2\}$.

$$\Pr\left(A_i \wedge \bigwedge_{j\in S_1} \overline{A_j} \,\middle|\, \bigwedge_{k\in S_2} \overline{A_k}\right) \le \Pr\left(A_i \,\middle|\, \bigwedge_{k\in S_2} \overline{A_k}\right)$$
$$= \Pr(A_i)$$
$$\le x_i \prod_{j\in S_1} (1 - x_j).$$

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$$\begin{aligned}
\Pr\left(A_i \wedge \bigwedge_{j\in S_1} \overline{A_j} \,\middle|\, \bigwedge_{k\in S_2} \overline{A_k}\right) &\leq \Pr\left(A_i \,\middle|\, \bigwedge_{k\in S_2} \overline{A_k}\right) \\
&= \Pr\left(A_i\right) \\
&\leq x_i \prod_{j\in S_1} (1 - x_j).
\end{aligned}$$

Proof of Lovász Local Lemma, Version 2

For the denominator, let $S_1 = \{j_1, j_2, \ldots, j_r\}$ (if it is empty, the denominator is 1).

$$\Pr\left(\overline{A_{j_1}} \wedge \overline{A_{j_2}} \wedge \cdots \wedge \overline{A_{j_r}} \;\middle|\; \bigwedge_{k \in S_2} \overline{A_k}\right)$$

$$= \Pr\left(\overline{A_{j_1}} \;\middle|\; \bigwedge_{k \in S_2} \overline{A_k}\right) \times \Pr\left(\overline{A_{j_2}} \;\middle|\; \overline{A_{j_1}} \wedge \bigwedge_{k \in S_2} \overline{A_k}\right) \times$$

$$\cdots \times \Pr\left(\overline{A_{j_r}} \;\middle|\; \bigwedge_{i=1}^{r-1} \overline{A_{j_i}} \wedge \bigwedge_{k \in S_2} \overline{A_k}\right)$$

$$\geq (1 - x_{j_1})(1 - x_{j_2}) \cdots (1 - x_{j_r}),$$

where the inequality holds by the induction hypothesis.

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where the inequality holds by the induction hypothesis.

Proof of Lovász Local Lemma, Version 2

Combining the two bounds,

$$\Pr\left(A_i \,\middle|\, \bigwedge_{j \in S} \overline{A_j}\right) = \frac{\Pr\left(A_i \wedge \bigwedge_{j \in S_1} \overline{A_j} \,\middle|\, \bigwedge_{k \in S_2} \overline{A_k}\right)}{\Pr\left(\bigwedge_{j \in S_1} \overline{A_j} \,\middle|\, \bigwedge_{k \in S_2} \overline{A_k}\right)}$$

$$\leq \frac{x_i \prod_{j \in S_1}(1 - x_j)}{\prod_{j \in S_1}(1 - x_j)}$$

$$= x_i,$$

which proves the claim.

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Proof of Lovász Local Lemma, Version 2

Returning to the Lovász Local Lemma,

$$\Pr\left(\bigwedge_{i=1}^{n} \overline{A_i}\right)$$

$$= \Pr\left(\overline{A_1}\right) \times \Pr\left(\overline{A_2} \mid \overline{A_1}\right) \times \cdots \times \Pr\left(\overline{A_n} \,\middle|\, \bigwedge_{j=1}^{n-1} \overline{A_j}\right)$$

$$\geq \prod_{i=1}^{n} (1 - x_i)$$

$$> 0,$$

which completes the proof.

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&\geq\prod_{i=1}^{n}(1-x_i)\\
&>0,
\end{aligned}$$

which completes the proof.

More General Versions

Dependency Digraph G

- Each vertex i corresponds to an event A_i.
- Each event A_i is independent of the event algebra generated by its non-neighbors in G.

More General Versions

How does this change the proof?

(Proof of claim by induction on $|S|$.)
Set $S_1 = \{j \in S \mid (i,j) \in E(G)\}$ and $S_2 = S \setminus S_1$. Now,

$$\Pr\left(A_i \,\middle|\, \bigwedge_{j \in S} \overline{A_j}\right) = \frac{\Pr\left(A_i \wedge \bigwedge_{j \in S_1} \overline{A_j} \,\middle|\, \bigwedge_{k \in S_2} \overline{A_k}\right)}{\Pr\left(\bigwedge_{j \in S_1} \overline{A_j} \,\middle|\, \bigwedge_{k \in S_2} \overline{A_k}\right)}.$$

We will bound the numerator and denominator separately.

More General Versions

Lemma (Lovász Local Lemma, Version 3)

Let $\{A_i \mid i \in [n]\}$ *be a collection of events having a dependency digraph* G.
If there are real numbers $x_i \in [0, 1)$ *such that, for all* i,

$$\Pr(A_i) \leq x_i \prod_{(i,j) \in E(G)} (1 - x_j),$$

then

$$\Pr\left(\bigwedge_{i=1}^{n} \overline{A_i}\right) > 0.$$

More General Versions

Negative Dependency Digraph G

- Each vertex i corresponds to an event A_i.
- For each event A_i and any subset S of its non-neighbors in G,

$$\Pr\left(\bigwedge_{j\in S}\overline{A_j}\right)\neq 0\rightarrow \Pr\left(A_i\;\middle|\;\bigwedge_{j\in S}\overline{A_j}\right)\leq \Pr\left(A_i\right).$$

More General Verions

How does this change the proof?

For the numerator, use the fact that A_i is negative dependent of $\{A_k \mid k \in S_2\}$.

$$\Pr\left(A_i \wedge \bigwedge_{j \in S_1} \overline{A_j} \;\middle|\; \bigwedge_{k \in S_2} \overline{A_k}\right) \leq \Pr\left(A_i \;\middle|\; \bigwedge_{k \in S_2} \overline{A_k}\right)$$

$$\leq \Pr\left(A_i\right)$$

$$\leq x_i \prod_{j \in S_1} (1 - x_j).$$

More General Versions

Lemma (Lopsided Lovász Local Lemma, Erdős-Spencer)

Let $\{A_i \mid i \in [n]\}$ *be a collection of events having a negative dependency digraph* G*.*
If there are real numbers $x_i \in [0,1)$ *such that, for all* i*,*

$$\Pr(A_i) \leq x_i \prod_{(i,j)\in E(G)} (1 - x_j),$$

then

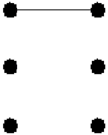
$$\Pr\left(\bigwedge_{i=1}^{n} \overline{A_i}\right) > 0.$$

Random Matchings in K_N

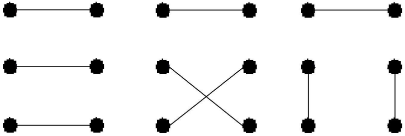
The canonical event for the matching M is

$$A_N^M = \{\text{perfect matchings } M' \text{ of } K_N \ (N \text{ even}) \mid M \subseteq M'\}.$$

M

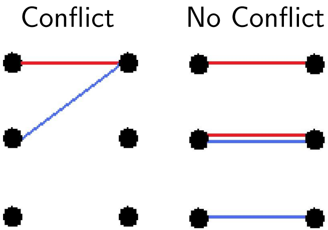


A_6^M



Random Matchings in K_N

Two matchings M_1 and M_2 conflict provided there are edges $e \in M_1$ and $f \in M_2$ such that $|e \cap f| = 1$.



Random Matchings in K_N

Lemma

For any collection $\mathcal{M}$ of matchings in K_N,

$$\Pr\left(\bigwedge_{M \in \mathcal{M}} A_N^M\right) \leq \Pr\left(\bigwedge_{M \in \mathcal{M}} A_{N+2}^M\right).$$

Random Matchings in K_N

Proof.

The proof relies on three observations about the space of random matchings.

Random Matchings in K_N

Observation One

Partition the set of all perfect matchings of K_{N+2} into

$$\mathcal{C}_i = \{\text{perfect matchings } M \mid i(N+2) \in M\}.$$

Notice

$$\Pr\left(\bigwedge_{M \in \mathcal{M}} \overline{A_{N+2}^M}\right) = \sum_{i=1}^{N+1} \Pr\left(\bigwedge_{M \in \mathcal{M}} \overline{A_{N+2}^M} \wedge \mathcal{C}_i\right).$$

Random Matchings in K_N

<u>Observation Two</u>

If M conflicts with $i(N+2)$, then $\mathcal{C}_i \subseteq \overline{A^M_{N+2}}$.
Letting $\mathcal{B}_i \subseteq \mathcal{M}$ be those matchings that do not conflict with $i(N+2)$, we can write

$$\bigwedge_{M \in \mathcal{M}} \overline{A^M_{N+2}} \wedge \mathcal{C}_i = \bigwedge_{M \in \mathcal{B}_i} \overline{A^M_{N+2}} \wedge \mathcal{C}_i.$$

Random Matchings in K_N

<u>Observation Three</u>

Consider any $[N+2] \to [N+2]$ function that interchanges i and $N+1$ and fixes all other elements.

Such a function

- stabilizes $\mathcal{B}_i$ (since $\mathcal{M}$ contains matchings of K_N),
- interchanges $\mathcal{C}_i$ and $\mathcal{C}_{N+1}$, and
- maps $\bigwedge_{M \in \mathcal{B}_i} \overline{A^M_{N+2}} \wedge \mathcal{C}_i$ to $\bigwedge_{M \in \mathcal{B}_i} \overline{A^M_{N+2}} \wedge \mathcal{C}_{N+1}$.

Random Matchings in K_N

$$
\begin{aligned}
\Pr\left(\bigwedge_{M\in\mathcal{M}} \overline{A^M_{N+2}}\right) &= \sum_{i=1}^{N+1} \Pr\left(\bigwedge_{M\in\mathcal{M}} \overline{A^M_{N+2}} \wedge \mathcal{C}_i\right) \\
&= \sum_{i=1}^{N+1} \Pr\left(\bigwedge_{M\in\mathcal{B}_i} \overline{A^M_{N+2}} \wedge \mathcal{C}_i\right) \\
&= \sum_{i=1}^{N+1} \Pr\left(\bigwedge_{M\in\mathcal{B}_i} \overline{A^M_{N+2}} \wedge \mathcal{C}_{N+1}\right) \\
&= \sum_{i=1}^{N+1} \Pr\left(\bigwedge_{M\in\mathcal{B}_i} \overline{A^M_{N+2}} \,\middle|\, \mathcal{C}_{N+1}\right) \Pr\left(\mathcal{C}_{N+1}\right) \\
&= \frac{1}{N+1} \sum_{i=1}^{N+1} \Pr\left(\bigwedge_{M\in\mathcal{B}_i} \overline{A^M_{N}}\right).
\end{aligned}
$$

Random Matchings in K_N

$$
\begin{aligned}
\Pr\left(\bigwedge_{M\in\mathcal{M}} \overline{A_{N+2}^{M}}\right) &= \sum_{i=1}^{N+1} \Pr\left(\bigwedge_{M\in\mathcal{M}} \overline{A_{N+2}^{M}} \wedge \mathcal{C}_i\right) \\
&= \sum_{i=1}^{N+1} \Pr\left(\bigwedge_{M\in\mathcal{B}_i} \overline{A_{N+2}^{M}} \wedge \mathcal{C}_i\right) \\
&= \sum_{i=1}^{N+1} \Pr\left(\bigwedge_{M\in\mathcal{B}_i} \overline{A_{N+2}^{M}} \wedge \mathcal{C}_{N+1}\right) \\
&= \sum_{i=1}^{N+1} \Pr\left(\bigwedge_{M\in\mathcal{B}_i} \overline{A_{N+2}^{M}} \,\middle|\, \mathcal{C}_{N+1}\right) \Pr\left(\mathcal{C}_{N+1}\right) \\
&= \frac{1}{N+1} \sum_{i=1}^{N+1} \Pr\left(\bigwedge_{M\in\mathcal{B}_i} \overline{A_{N}^{M}}\right).
\end{aligned}
$$

Random Matchings in K_N

$$\Pr\left(\bigwedge_{M\in\mathcal{M}}\overline{A^M_{N+2}}\right)=\sum_{i=1}^{N+1}\Pr\left(\bigwedge_{M\in\mathcal{M}}\overline{A^M_{N+2}}\wedge\mathcal{C}_i\right)$$

$$=\sum_{i=1}^{N+1}\Pr\left(\bigwedge_{M\in\mathcal{B}_i}\overline{A^M_{N+2}}\wedge\mathcal{C}_i\right)$$

$$=\sum_{i=1}^{N+1}\Pr\left(\bigwedge_{M\in\mathcal{B}_i}\overline{A^M_{N+2}}\wedge\mathcal{C}_{N+1}\right)$$

$$=\sum_{i=1}^{N+1}\Pr\left(\bigwedge_{M\in\mathcal{B}_i}\overline{A^M_{N+2}}\,\middle|\,\mathcal{C}_{N+1}\right)\Pr\left(\mathcal{C}_{N+1}\right)$$

$$=\frac{1}{N+1}\sum_{i=1}^{N+1}\Pr\left(\bigwedge_{M\in\mathcal{B}_i}\overline{A^M_{N}}\right).$$

Random Matchings in K_N

$$
\begin{aligned}
\Pr\left(\bigwedge_{M\in\mathcal{M}} \overline{A_{N+2}^M}\right) &= \sum_{i=1}^{N+1} \Pr\left(\bigwedge_{M\in\mathcal{M}} \overline{A_{N+2}^M} \wedge \mathcal{C}_i\right) \\
&= \sum_{i=1}^{N+1} \Pr\left(\bigwedge_{M\in\mathcal{B}_i} \overline{A_{N+2}^M} \wedge \mathcal{C}_i\right) \\
&= \sum_{i=1}^{N+1} \Pr\left(\bigwedge_{M\in\mathcal{B}_i} \overline{A_{N+2}^M} \wedge \mathcal{C}_{N+1}\right) \\
&= \sum_{i=1}^{N+1} \Pr\left(\bigwedge_{M\in\mathcal{B}_i} \overline{A_{N+2}^M} \,\middle|\, \mathcal{C}_{N+1}\right) \Pr\left(\mathcal{C}_{N+1}\right) \\
&= \frac{1}{N+1} \sum_{i=1}^{N+1} \Pr\left(\bigwedge_{M\in\mathcal{B}_i} \overline{A_N^M}\right).
\end{aligned}
$$

Random Matchings in K_N

$$\Pr\left(\bigwedge_{M\in\mathcal{M}} \overline{A_{N+2}^M}\right) = \sum_{i=1}^{N+1} \Pr\left(\bigwedge_{M\in\mathcal{M}} \overline{A_{N+2}^M} \wedge \mathcal{C}_i\right)$$

$$= \sum_{i=1}^{N+1} \Pr\left(\bigwedge_{M\in\mathcal{B}_i} \overline{A_{N+2}^M} \wedge \mathcal{C}_i\right)$$

$$= \sum_{i=1}^{N+1} \Pr\left(\bigwedge_{M\in\mathcal{B}_i} \overline{A_{N+2}^M} \wedge \mathcal{C}_{N+1}\right)$$

$$= \sum_{i=1}^{N+1} \Pr\left(\bigwedge_{M\in\mathcal{B}_i} \overline{A_{N+2}^M} \,\middle|\, \mathcal{C}_{N+1}\right) \Pr\left(\mathcal{C}_{N+1}\right)$$

$$= \frac{1}{N+1} \sum_{i=1}^{N+1} \Pr\left(\bigwedge_{M\in\mathcal{B}_i} \overline{A_N^M}\right).$$

Random Matchings in K_N

$$\begin{aligned}
\Pr\left(\bigwedge_{M\in\mathcal{M}} \overline{A_{N+2}^{M}}\right) &= \sum_{i=1}^{N+1} \Pr\left(\bigwedge_{M\in\mathcal{M}} \overline{A_{N+2}^{M}} \wedge \mathcal{C}_i\right) \\
&= \sum_{i=1}^{N+1} \Pr\left(\bigwedge_{M\in\mathcal{B}_i} \overline{A_{N+2}^{M}} \wedge \mathcal{C}_i\right) \\
&= \sum_{i=1}^{N+1} \Pr\left(\bigwedge_{M\in\mathcal{B}_i} \overline{A_{N+2}^{M}} \wedge \mathcal{C}_{N+1}\right) \\
&= \sum_{i=1}^{N+1} \Pr\left(\bigwedge_{M\in\mathcal{B}_i} \overline{A_{N+2}^{M}} \;\middle|\; \mathcal{C}_{N+1}\right) \Pr\left(\mathcal{C}_{N+1}\right) \\
&= \frac{1}{N+1} \sum_{i=1}^{N+1} \Pr\left(\bigwedge_{M\in\mathcal{B}_i} \overline{A_{N}^{M}}\right).
\end{aligned}$$

Random Matchings in K_N

Continuing from before,

$$\Pr\left(\bigwedge_{M\in\mathcal{M}} \overline{A^M_{N+2}}\right) = \frac{1}{N+1}\sum_{i=1}^{N+1} \Pr\left(\bigwedge_{M\in\mathcal{B}_i} \overline{A^M_N}\right)$$

$$\geq \frac{1}{N+1}\sum_{i=1}^{N+1} \Pr\left(\bigwedge_{M\in\mathcal{M}} \overline{A^M_N}\right)$$

$$= \frac{1}{N+1}(N+1)\Pr\left(\bigwedge_{M\in\mathcal{M}} \overline{A^M_N}\right)$$

$$= \Pr\left(\bigwedge_{M\in\mathcal{M}} \overline{A^M_N}\right),$$

which proves the lemma.

Random Matchings in K_N

Continuing from before,

$$\begin{aligned}
\Pr\left(\bigwedge_{M\in\mathcal{M}} \overline{A_{N+2}^{M}}\right) &= \frac{1}{N+1}\sum_{i=1}^{N+1} \Pr\left(\bigwedge_{M\in\mathcal{B}_i} \overline{A_{N}^{M}}\right) \\
&\geq \frac{1}{N+1}\sum_{i=1}^{N+1} \Pr\left(\bigwedge_{M\in\mathcal{M}} \overline{A_{N}^{M}}\right) \\
&= \frac{1}{N+1}(N+1)\Pr\left(\bigwedge_{M\in\mathcal{M}} \overline{A_{N}^{M}}\right) \\
&= \Pr\left(\bigwedge_{M\in\mathcal{M}} \overline{A_{N}^{M}}\right),
\end{aligned}$$

which proves the lemma.

Continuing from before,

$$\Pr\left(\bigwedge_{M\in\mathcal{M}} \overline{A_{N+2}^M}\right) = \frac{1}{N+1}\sum_{i=1}^{N+1}\Pr\left(\bigwedge_{M\in\mathcal{B}_i} \overline{A_N^M}\right)$$
$$\geq \frac{1}{N+1}\sum_{i=1}^{N+1}\Pr\left(\bigwedge_{M\in\mathcal{M}} \overline{A_N^M}\right)$$
$$= \frac{1}{N+1}(N+1)\Pr\left(\bigwedge_{M\in\mathcal{M}} \overline{A_N^M}\right)$$
$$= \Pr\left(\bigwedge_{M\in\mathcal{M}} \overline{A_N^M}\right),$$

which proves the lemma.

Random Matchings in K_N

Continuing from before,

$$\begin{aligned}\Pr\left(\bigwedge_{M\in\mathcal{M}}\overline{A^M_{N+2}}\right) &= \frac{1}{N+1}\sum_{i=1}^{N+1}\Pr\left(\bigwedge_{M\in\mathcal{B}_i}\overline{A^M_N}\right)\\ &\geq \frac{1}{N+1}\sum_{i=1}^{N+1}\Pr\left(\bigwedge_{M\in\mathcal{M}}\overline{A^M_N}\right)\\ &= \frac{1}{N+1}(N+1)\Pr\left(\bigwedge_{M\in\mathcal{M}}\overline{A^M_N}\right)\\ &= \Pr\left(\bigwedge_{M\in\mathcal{M}}\overline{A^M_N}\right),\end{aligned}$$

which proves the lemma.

Random Matchings in K_N

Continuing from before,

$$\Pr\left(\bigwedge_{M\in\mathcal{M}} \overline{A_{N+2}^M}\right) = \frac{1}{N+1}\sum_{i=1}^{N+1} \Pr\left(\bigwedge_{M\in\mathcal{B}_i} \overline{A_N^M}\right)$$

$$\geq \frac{1}{N+1}\sum_{i=1}^{N+1} \Pr\left(\bigwedge_{M\in\mathcal{M}} \overline{A_N^M}\right)$$

$$= \frac{1}{N+1}(N+1)\Pr\left(\bigwedge_{M\in\mathcal{M}} \overline{A_N^M}\right)$$

$$= \Pr\left(\bigwedge_{M\in\mathcal{M}} \overline{A_N^M}\right),$$

which proves the lemma.

Random Matchings in K_N

The conflict graph G for a collection of events $\{A_i \mid i \in [n]\}$ is the graph on $[n]$ with

$$E(G) = \{ij \mid A_i \text{ and } A_j \text{ conflict}\}.$$

Random Matchings in K_N

Theorem (Lu-Székely)

Let $\mathcal{M}$ be a collection of matchings in K_N. The conflict graph for the collection of canonical events $\{A^M \mid M \in \mathcal{M}\}$ is a negative dependency graph.

Proof.

Fix a matching $M_0 \in \mathcal{M}$. Let $\mathcal{S} \subseteq \mathcal{M}$ be such that M_0 does not conflict with any $M \in \mathcal{S}$. We are to show

$$\Pr\left(A^{M_0} \;\middle|\; \bigwedge_{M \in \mathcal{S}} \overline{A^M}\right) \leq \Pr\left(A^{M_0}\right),$$

which is equivalent to

$$\Pr\left(\bigwedge_{M \in \mathcal{S}} \overline{A^M} \;\middle|\; A^{M_0}\right) \leq \Pr\left(\bigwedge_{M \in S} \overline{A^M}\right).$$

Random Matchings in K_N

Define

$$\mathcal{S}' = \{M \setminus M_0 \mid M \in \mathcal{S}\}.$$

If $\emptyset \in \mathcal{S}'$, then

$$\Pr\left(\bigwedge_{M \in \mathcal{S}} \overline{A^M} \,\middle|\, A^{M_0}\right) = 0,$$

and we are finished.

Random Matchings in K_N

Otherwise, let π be any permutation mapping $V(M_0)$ to $\{N - |M_0| + 1, \ldots, N\}$.
The matchings in the collection $\mathcal{S}'$ are all vertex-disjoint from M_0, so the matchings of $\pi(\mathcal{S}')$ are contained in $K_{N-|M_0|}$.

Random Matchings in K_N

$$
\begin{aligned}
\Pr\left(\bigwedge_{M\in\mathcal{S}} \overline{A_N^M} \,\middle|\, A_N^{M_0}\right) &= \Pr\left(\bigwedge_{M\in\mathcal{S}} \overline{A_N^{M\setminus M_0}} \,\middle|\, A_N^{M_0}\right) \\
&= \Pr\left(\bigwedge_{M\in\mathcal{S}} \overline{A_N^{\pi(M\setminus M_0)}} \,\middle|\, A_N^{\pi(M_0)}\right) \\
&= \Pr\left(\bigwedge_{M\in\mathcal{S}} \overline{A_{N-|M_0|}^{\pi(M\setminus M_0)}}\right) \\
&\leq \Pr\left(\bigwedge_{M\in\mathcal{S}} \overline{A_N^{\pi(M\setminus M_0)}}\right) \\
&= \Pr\left(\bigwedge_{M\in\mathcal{S}} \overline{A_N^{M\setminus M_0}}\right) \\
&\leq \Pr\left(\bigwedge_{M\in\mathcal{S}} \overline{A_N^{M}}\right). \qquad \square
\end{aligned}
$$

Random Matchings in K_N

$$\Pr\left(\bigwedge_{M\in\mathcal{S}} \overline{A_N^M} \;\middle|\; A_N^{M_0}\right) = \Pr\left(\bigwedge_{M\in\mathcal{S}} \overline{A_N^{M\setminus M_0}} \;\middle|\; A_N^{M_0}\right)$$

$$= \Pr\left(\bigwedge_{M\in\mathcal{S}} \overline{A_N^{\pi(M\setminus M_0)}} \;\middle|\; A_N^{\pi(M_0)}\right)$$

$$= \Pr\left(\bigwedge_{M\in\mathcal{S}} \overline{A_{N-|M_0|}^{\pi(M\setminus M_0)}}\right)$$

$$\leq \Pr\left(\bigwedge_{M\in\mathcal{S}} \overline{A_N^{\pi(M\setminus M_0)}}\right)$$

$$= \Pr\left(\bigwedge_{M\in\mathcal{S}} \overline{A_N^{M\setminus M_0}}\right)$$

$$\leq \Pr\left(\bigwedge_{M\in\mathcal{S}} \overline{A_N^M}\right). \qquad \square$$

Random Matchings in K_N

$$
\begin{aligned}
\Pr\left(\bigwedge_{M\in\mathcal{S}} \overline{A_N^{M}} \;\middle|\; A_N^{M_0}\right) &= \Pr\left(\bigwedge_{M\in\mathcal{S}} \overline{A_N^{M\setminus M_0}} \;\middle|\; A_N^{M_0}\right) \\
&= \Pr\left(\bigwedge_{M\in\mathcal{S}} \overline{A_N^{\pi(M\setminus M_0)}} \;\middle|\; A_N^{\pi(M_0)}\right) \\
&= \Pr\left(\bigwedge_{M\in\mathcal{S}} \overline{A_{N-|M_0|}^{\pi(M\setminus M_0)}}\right) \\
&\le \Pr\left(\bigwedge_{M\in\mathcal{S}} \overline{A_N^{\pi(M\setminus M_0)}}\right) \\
&= \Pr\left(\bigwedge_{M\in\mathcal{S}} \overline{A_N^{M\setminus M_0}}\right) \\
&\le \Pr\left(\bigwedge_{M\in\mathcal{S}} \overline{A_N^{M}}\right). \qquad \square
\end{aligned}
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Random Matchings in K_N

$$
\begin{aligned}
\Pr\left(\bigwedge_{M\in\mathcal{S}} \overline{A_N^M} \;\middle|\; A_N^{M_0}\right) &= \Pr\left(\bigwedge_{M\in\mathcal{S}} \overline{A_N^{M\setminus M_0}} \;\middle|\; A_N^{M_0}\right) \\
&= \Pr\left(\bigwedge_{M\in\mathcal{S}} \overline{A_N^{\pi(M\setminus M_0)}} \;\middle|\; A_N^{\pi(M_0)}\right) \\
&= \Pr\left(\bigwedge_{M\in\mathcal{S}} \overline{A_{N-|M_0|}^{\pi(M\setminus M_0)}}\right) \\
&\leq \Pr\left(\bigwedge_{M\in\mathcal{S}} \overline{A_N^{\pi(M\setminus M_0)}}\right) \\
&= \Pr\left(\bigwedge_{M\in\mathcal{S}} \overline{A_N^{M\setminus M_0}}\right) \\
&\leq \Pr\left(\bigwedge_{M\in\mathcal{S}} \overline{A_N^{M}}\right). \qquad \square
\end{aligned}
$$

Random Matchings in K_N

$$\begin{aligned}
\Pr\left(\bigwedge_{M\in\mathcal{S}} \overline{A_N^M} \,\middle|\, A_N^{M_0}\right) &= \Pr\left(\bigwedge_{M\in\mathcal{S}} \overline{A_N^{M\setminus M_0}} \,\middle|\, A_N^{M_0}\right) \\
&= \Pr\left(\bigwedge_{M\in\mathcal{S}} \overline{A_N^{\pi(M\setminus M_0)}} \,\middle|\, A_N^{\pi(M_0)}\right) \\
&= \Pr\left(\bigwedge_{M\in\mathcal{S}} \overline{A_{N-|M_0|}^{\pi(M\setminus M_0)}}\right) \\
&\leq \Pr\left(\bigwedge_{M\in\mathcal{S}} \overline{A_N^{\pi(M\setminus M_0)}}\right) \\
&= \Pr\left(\bigwedge_{M\in\mathcal{S}} \overline{A_N^{M\setminus M_0}}\right) \\
&\leq \Pr\left(\bigwedge_{M\in\mathcal{S}} \overline{A_N^{M}}\right). \qquad \square
\end{aligned}$$

Random Matchings in K_N

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\begin{aligned}
\Pr\left(\bigwedge_{M\in\mathcal{S}} \overline{A_N^M} \;\middle|\; A_N^{M_0}\right) &= \Pr\left(\bigwedge_{M\in\mathcal{S}} \overline{A_N^{M\setminus M_0}} \;\middle|\; A_N^{M_0}\right) \\
&= \Pr\left(\bigwedge_{M\in\mathcal{S}} \overline{A_N^{\pi(M\setminus M_0)}} \;\middle|\; A_N^{\pi(M_0)}\right) \\
&= \Pr\left(\bigwedge_{M\in\mathcal{S}} \overline{A_{N-|M_0|}^{\pi(M\setminus M_0)}}\right) \\
&\le \Pr\left(\bigwedge_{M\in\mathcal{S}} \overline{A_N^{\pi(M\setminus M_0)}}\right) \\
&= \Pr\left(\bigwedge_{M\in\mathcal{S}} \overline{A_N^{M\setminus M_0}}\right) \\
&\le \Pr\left(\bigwedge_{M\in\mathcal{S}} \overline{A_N^{M}}\right). \qquad \square
\end{aligned}
$$

Random Matchings in K_N

$$
\begin{aligned}
\Pr\left(\bigwedge_{M\in\mathcal{S}} \overline{A_N^M} \;\middle|\; A_N^{M_0}\right) &= \Pr\left(\bigwedge_{M\in\mathcal{S}} \overline{A_N^{M\setminus M_0}} \;\middle|\; A_N^{M_0}\right) \\
&= \Pr\left(\bigwedge_{M\in\mathcal{S}} \overline{A_N^{\pi(M\setminus M_0)}} \;\middle|\; A_N^{\pi(M_0)}\right) \\
&= \Pr\left(\bigwedge_{M\in\mathcal{S}} \overline{A_{N-|M_0|}^{\pi(M\setminus M_0)}}\right) \\
&\leq \Pr\left(\bigwedge_{M\in\mathcal{S}} \overline{A_N^{\pi(M\setminus M_0)}}\right) \\
&= \Pr\left(\bigwedge_{M\in\mathcal{S}} \overline{A_N^{M\setminus M_0}}\right) \\
&\leq \Pr\left(\bigwedge_{M\in\mathcal{S}} \overline{A_N^M}\right).
\end{aligned}
$$

□

Random Matchings in K_N

$$
\begin{aligned}
\Pr\left(\bigwedge_{M\in\mathcal{S}} \overline{A_N^M} \;\middle|\; A_N^{M_0}\right) &= \Pr\left(\bigwedge_{M\in\mathcal{S}} \overline{A_N^{M\setminus M_0}} \;\middle|\; A_N^{M_0}\right) \\
&= \Pr\left(\bigwedge_{M\in\mathcal{S}} \overline{A_N^{\pi(M\setminus M_0)}} \;\middle|\; A_N^{\pi(M_0)}\right) \\
&= \Pr\left(\bigwedge_{M\in\mathcal{S}} \overline{A_{N-|M_0|}^{\pi(M\setminus M_0)}}\right) \\
&\leq \Pr\left(\bigwedge_{M\in\mathcal{S}} \overline{A_N^{\pi(M\setminus M_0)}}\right) \\
&= \Pr\left(\bigwedge_{M\in\mathcal{S}} \overline{A_N^{M\setminus M_0}}\right) \\
&\leq \Pr\left(\bigwedge_{M\in\mathcal{S}} \overline{A_N^M}\right). \qquad \square
\end{aligned}
$$

Random Matchings in Hypergraphs

A modification of the proof gives a hypergraph version of the result.

Theorem (Lu-M.-Székely)

Let $\mathcal{M}$ be a collection of matchings in K_N^k. The conflict graph for the collection of canonical events $\{A^M \mid M \in \mathcal{M}\}$ is a negative dependency graph.

Thanks

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Further Details

L. Lu and L.A. Székely

A New Asymptotic Enumeration Technique: The Lovász Local Lemma