Math 731 Homework 9

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March 29, 2010

1 Extra Problem 1

Lemma 1.1. *Every point of a closed ball in an ultrametric space is a center.*

Proof. Let (X, ρ) be an ultrametric space, and let $B(a, \epsilon)$ denote the closed ball around $a \in X$ of radius $\epsilon > 0$ (i.e. the set $\{y \in X \mid \rho(a, y) \leq \epsilon\}$). Let $b \in B(a, \epsilon)$ and let $x \in B(b, \epsilon)$. It follows that

$$\begin{aligned} \rho(a, x) &\leq \max\{\rho(a, b), \rho(b, x)\} \\ &\leq \epsilon, \end{aligned}$$

and so $x \in B(a, \epsilon)$. Hence, $B(b, \epsilon) \subset B(a, \epsilon)$, and a similar proof gives the reverse inclusion. Therefore, $B(a, \epsilon) = B(b, \epsilon)$, and so both a and b are centers. □

Proposition 1.2. *Every ultrametric space has a base of clopen sets.*

Proof. Let X be an ultrametric space and let $\mathcal{F}$ denote the collection of all closed balls in X. Evidently, $\mathcal{F}$ covers X. Observe next that, for any two balls of $\mathcal{F}$, either they are disjoint or one is contained in the other. To see this, let $F_1, F_2 \in \mathcal{F}$ with $x \in F_1 \cap F_2$ (such an x exists if F_1 and F_2 are not disjoint). Now, F_1 has the form $B(a, \epsilon_1)$ and F_2 has the form $B(b, \epsilon_2)$. Without loss of generality, let $\epsilon_1 \leq \epsilon_2$. By the lemma, $F_1 = B(a, \epsilon_1)$ and $F_2 = B(a, \epsilon_2)$, and so $F_1 \subset F_2$. From this fact, we conclude that $\mathcal{F}$ is a base for X (any F_1 and F_2 with nontrivial intersection has one containing the other).

Now, closed balls are certainly closed in this topology, as $\overline{F} = F$ for all $F \in \mathcal{F}$. To see that they are also open, observe that, by the lemma, F contains a basic neighborhood of each of its points (namely, F itself). Therefore, $\mathcal{F}$ is a base of clopen sets for X. □

2 Extra Problem 2

Proposition 2.1. *The metric given by*

$$d(x,y) = \begin{cases} 0 & \text{if } x = y \\ \frac{1}{2^n} & \text{otherwise, where } n = \max\{n \mid x(i) = y(i) \, \forall i \leq n\} \end{cases}$$

gives the product topology on $\mathbb{N}^{\mathbb{N}}$.

Proof. Let $\mathcal{B}$ denote the collection of all open balls of $\mathbb{N}^{\mathbb{N}}$. Evidently, $\mathcal{B}$ covers $\mathbb{N}^{\mathbb{N}}$. To see that it is, in fact, a base for $\mathbb{N}^{\mathbb{N}}$, let $B_1, B_2 \in \mathcal{B}$ and let $x \in B_1 \cap B_2$. Now, B_1 is of the form $B(a, \frac{1}{2^m})$ and B_2 is of the form $B(b, \frac{1}{2^n})$. Without loss of generality, let $m \leq n$. It follows that $x(i) = a(i) = b(i)$ for all $i \leq m$. Hence, $x \in B(x, \frac{1}{2^m}) \subset B_1 \cap B_2$, and so $\mathcal{B}$ is a base for $\mathbb{N}^{\mathbb{N}}$.

Now, the sets of $\mathcal{B}$ can be expressed as

$$B\left(x, \frac{1}{2^n}\right) = \prod_{i=1}^{n} \{x(i)\} \times \prod_{i=n+1}^{\infty} \mathbb{N}.$$

In other words, the basic open sets are products of open sets U_n of $\mathbb{N}$ where $U_n = \mathbb{N}$ for all but finitely-many n. Hence, $\mathcal{B}$ induces the product topology on $\mathbb{N}^{\mathbb{N}}$. □