Math 704 Homework 9

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Problem 1

Suppose f is defined on $\mathbb{R}^2$ as follows

$$f(x,y) = \begin{cases} a_n & \text{if } n \geq 0, n \leq x < n+1, n \leq y < n+1 \\ -a_n & \text{if } n \geq 0, n \leq x < n+1, n+1 \leq y < n+2 \\ 0 & \text{otherwise} \end{cases}$$

Here $a_n = \displaystyle\sum_{k \leq n} b_k$ with $\{b_k\}$ a positive sequence such that $\displaystyle\sum_{k=0}^{\infty} b_k = s < \infty$.

a. Verify that each slice f^y and f_x is integrable. Also, for all x, $\int_{\mathbb{R}} f_x(y)dy = 0$, and hence $\int \left(\int_{\mathbb{R}} f(x,y)dy \right) dx = 0$.

Proof. Observe that for $0 \leq y < 1$

$$f^y(x) = \begin{cases} a_0 & \text{if } 0 \leq x < 1 \\ 0 & \text{otherwise} \end{cases}$$

and for $n \leq y < n+1$ with $n \geq 1$

$$f^y(x) = \begin{cases} a_n & \text{if } n \leq x < n+1 \\ -a_{n-1} & \text{if } n-1 \leq x < n \\ 0 & \text{otherwise} \end{cases}$$

Also, for all $n \geq 0$

$$f_x(y) = \begin{cases} a_n & \text{if } n \leq y < n+1 \\ -a_n & \text{if } n+1 \leq y < n+2 \\ 0 & \text{otherwise} \end{cases}$$

all of which are step functions, and hence measurable. We will compute the values of their integrals as they are needed. These values will turn out to be finite, and so the slices will be shown integrable.

Let $n \leq x < n+1$. We have

$$\begin{aligned} \int_{\mathbb{R}} f_x(y)dy &= \int_{[n,n+1]} a_n dy + \int_{[n+1,n+2]} -a_n dy \\ &= a_n - a_n \\ &= 0 \end{aligned}$$

and so f_x is integrable and

$$\begin{aligned} \int_{\mathbb{R}} \left(\int_{\mathbb{R}} f(x,y)dy \right) dx &= \int_{\mathbb{R}} \left(\int_{\mathbb{R}} f_x(y)dy \right) dx \\ &= \int_{\mathbb{R}} 0dx \\ &= 0 \end{aligned}$$

□

b. However,

$$\int_{\mathbb{R}} f^y(x)dy = \begin{cases} a_0 & \text{if } 0 \leq y < 1 \\ a_n - a_{n-1} & \text{if } n \geq 1, n \leq y < n+1 \end{cases}$$

Hence, $y \mapsto \int_{\mathbb{R}} f^y(x)dx$ is integrable on $(0, \infty)$ and

$$\int_{\mathbb{R}} \left(\int_{\mathbb{R}} f(x,y)dx \right) dy = s$$

Proof. Let $0 \leq y < 1$. From our definition of $f^y(x)$ in part (a), we have

$$\begin{aligned} \int_{\mathbb{R}} f^y(x)dx &= \int_{[0,1]} a_0 dx \\ &= a_0 \end{aligned}$$

Similarly, if $n \leq y < n+1$ with $n \geq 1$, we have

$$\begin{aligned} \int_{\mathbb{R}} f^y(x)dx &= \int_{[n-1,n]} -a_{n-1}dx + \int_{[n,n+1]} a_n dx \\ &= a_n - a_{n-1} \end{aligned}$$

Hence, for every y, f^y is integrable. Now, Tonelli's Theorem gives that $y \mapsto \int_{\mathbb{R}} f^y(x)dx$ is measurable. Furthermore,

$$\begin{aligned} \int_{\mathbb{R}} \left(\int_{\mathbb{R}} f(x,y)dx \right) dy &= \int_{\mathbb{R}} \left(\int_{\mathbb{R}} f^y(x)dx \right) dy \\ &= \sum_{n=0}^{\infty} \left(\int_{[n,n+1]} \left(\int_{\mathbb{R}} f^y(x)dx \right) dy \right) \\ &= \sum_{n=0}^{\infty} \left(\int_{[n,n+1]} (a_n - a_{n-1})\, dy \right) \\ &= \sum_{n=0}^{\infty} (a_n - a_{n-1}) \\ &= s \end{aligned}$$

where we define the term a_{-1} to be 0. □

c. Note that $\int_{\mathbb{R}\times\mathbb{R}} |f(x,y)|dxdy = \infty$.

Proof.

$$\begin{aligned} \int_{\mathbb{R}\times\mathbb{R}} |f(x,y)|dxdy &= \int_{\mathbb{R}} \left(\int_{\mathbb{R}} |f(x,y)|dx \right) dy \\ &= \int_{\mathbb{R}} \left(\int_{(0,\infty)} |f(x,y)|dx \right) dy \\ &\geq \int_{\mathbb{R}} \left(\int_{(0,\infty)} a_0 dx \right) dy & \text{(since } \{a_n\} \text{ is a monotone increasing sequence)} \\ &= \infty \end{aligned}$$

□

Problem 2

Suppose f is integrable on $\mathbb{R}^d$. For each $\alpha > 0$, let $E_\alpha = \{x : |f(x)| > \alpha\}$. Prove that

$$\int_{\mathbb{R}^d} |f(x)|dx = \int_0^\infty m(E_\alpha)d\alpha$$

Proof. Observe first that, for fixed $\alpha > 0$,

$$\begin{aligned} E_\alpha &= \{x : |f(x)| > \alpha\} \\ &= \{x : \alpha > f(x) > -\alpha\} \end{aligned}$$

Since f is integrable, it is measurable, and so E_α is measurable. Now,

$$\begin{aligned} \int_0^\infty m(E_\alpha)d\alpha &= \int_0^\infty \left(\int_{\mathbb{R}^d} \chi_{E_\alpha}(t)dt \right) d\alpha \\ &= \int_{\mathbb{R}^d} \left(\int_0^\infty \chi_{E_\alpha}(t)d\alpha \right) dt && \text{(by Tonelli's Theorem)} \\ &= \int_{\mathbb{R}^d} \left(\int_0^{|f(t)|} 1d\alpha \right) dt && \text{(since } \chi_{E_\alpha} = 0 \text{ for } \alpha \geq |f(t)| \text{ for fixed } t) \\ &= \int_{\mathbb{R}^d} |f(t)|dt \end{aligned}$$

□

Problem 3

Consider the convolution

$$(f * g)(x) = \int_{\mathbb{R}^d} f(x-y)g(y)\, dy.$$

a. Show that $f * g$ is uniformly continuous when f is integrable and g bounded.

Proof. Let $g(x) \leq M$ for all x. We have

$$\left| \int_{\mathbb{R}}^d f(x-y)g(y)\, dy \right| \leq M \int_{\mathbb{R}}^d |f(x-y)|\, dy < \infty$$

Let $\epsilon > 0$. We know $\|f_h - f\|_1 \to 0$ as $h \to 0$ (from a previous homework). Hence, we can find $\delta > 0$ such that $\|f_h - f\|_1 < \frac{\epsilon}{M}$ whenever $h < \delta$. Now

$$\begin{aligned} |(f*g)(x_1) - (f*g)(x_2)| &= \left| \int_{\mathbb{R}}^d f(x_1 - y)g(y)dy - \int_{\mathbb{R}}^d f(x_2 - y)g(y)dy \right| \\ &= \left| \int_{\mathbb{R}}^d (f(x_1 - y) - f(x_2 - y))g(y)dy \right| \end{aligned}$$

Perform the change of variable $u = x_2 - y$ to get

$$\begin{aligned} \left| \int_{\mathbb{R}}^d (f(u - (x_1 - x_2)) - f(u))g(-u - x_2)du \right| &\leq \int_{\mathbb{R}}^d |f(u - (x_1 - x_2)) - f(u)||g(-u - x_2)|du \\ &\leq M \int_{\mathbb{R}}^d |f(u - (x_1 - x_2)) - f(u)|du \\ &= M\|f_{x_1 - x_2} - f\|_1 \end{aligned}$$

Now, $\|f_{x_1-x_2} - f\|_1 < \frac{\epsilon}{M}$ whenever $|x_1 - x_2| < \delta$. Since δ was arbitrary, we see that $f * g$ is uniformly continuous. □

b. If in addition g is integrable, prove that $(f * g)(x) \to 0$ as $|x| \to \infty$.

Proof. Since both f and g are integrable, we have that $f * g$ is integrable. By a previous homework, we know that an uniformly continuous, integrable function tends to 0. Therefore, $(f * g)(x) \to 0$ as $|x| \to \infty$. $\square$

Problem 4

Let $E \subset [0,1] \times [0,1]$ be a measurable set. Assume that $m(E_x) \leq \frac{1}{2}$ for almost every $x \in [0,1]$. Prove that $m(\{y \in [0,1] \mid m(E^y) = 1\}) \leq \frac{1}{2}$.

Proof. First, let F denote the subset of $[0,1]$ where $m(E_x) \leq \frac{1}{2}$. We have that $m(F) = 1$. By a corollary of Tonelli's Theorem, we know that

$$\begin{aligned} m(E) &= \int_{[0,1]} m(E_x)dx \\ &= \int_F m(E_x)dx \\ &\leq \int_F \frac{1}{2}dx \\ &= m(F) \cdot \frac{1}{2} \\ &= \frac{1}{2} \end{aligned}$$

Next, let G denote the set $\{y \in [0,1] \mid m(E^y) = 1\}$. By the same corollary as before, we have

$$\begin{aligned} m(E) &= \int_{[0,1]} m(E^y)dy \\ &\geq \int_G m(E^y)dy \\ &= \int_G 1dy \\ &= m(G) \end{aligned}$$

From the previous observation, we know that $m(E) \leq \frac{1}{2}$. Therefore, $m(G) \leq \frac{1}{2}$, as desired. $\square$

Problem 5

Let $f \in L^1(\mathbb{R})$ and define for $h > 0$

$$\phi_h(x) = \tfrac{1}{2h} \int_{x-h}^{x+h} f(t)dt$$

Prove that ϕ_h is integrable and $||\phi_h||_1 \leq ||f||_1$.

Proof. Define $F(t,x) : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ so that

$$F(t,x) = f(t)\chi_A(t,x), \text{ where } A = \{(t,x) \mid x - h \leq t < x + h\}$$

Now, A is the intersection of a closed half-plane and an open half-plane, so A is measurable. Furthermore, $f \in L^1$, so f is measurable. Taken together, we have that $F(t,x)$ is measurable. Now

$$\textstyle\int_{\mathbb{R}} F(t,x)dt = \int_{x-h}^{x+h} f(t)dt < \infty$$

and so

$$
\begin{aligned}
||\phi_h||_1 &= \int_{\mathbb{R}} |\phi(x)|dx \\
&= \int_{\mathbb{R}} |\frac{1}{2h} \int_{x-h}^{x+h} f(t)dt|dx \\
&= \int_{\mathbb{R}} |\frac{1}{2h} \int_{\mathbb{R}} F(t,x)dt|dx \\
&\leq \frac{1}{2h}(\int_{\mathbb{R}} \int_{\mathbb{R}} |F(t,x)|dt)dx \\
&= \frac{1}{2h}(\int_{\mathbb{R}} \int_{\mathbb{R}} |F(t,x)|dx)dt && \text{(by Fubini's Theorem)} \\
&= \frac{1}{2h} \int_{\mathbb{R}} 2h|f(t)|dt && \text{(since } F(t,x) = 0 \text{ for } t \notin [x-h, x+h)) \\
&= \int_{\mathbb{R}} |f(t)|dt \\
&= ||f||_1
\end{aligned}
$$

□