Math 704 Homework 8

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<u>Problem 1</u>

Consider the function defined over $\mathbb{R}$ by

$$f(x) = \begin{cases} x^{-1/2} & \text{if } 0 < x < 1, \\ 0 & \text{otherwise} \end{cases}$$

For a fixed enumeration $\{r_n\}_{n=1}^{\infty}$ of the rationals $\mathbb{Q}$, let

$$F(x) = \sum_{n=1}^{\infty} 2^{-n} f(x - r_n).$$

Prove that F is integrable, hence the series defining F converges for almost every $x \in \mathbb{R}$. However, observe that this series is unbounded on every interval, and in fact, any function $\tilde{F}$ that agrees with F almost everywhere is unbounded in any interval.

Proof. Define $s_k = \sum_{n=1}^{k} 2^{-n} f(x - r_n)$. Since 2^{-n} and $f(x - r_n)$ are both measurable for each n, we have that s_k is measurable for each k. Furthermore, $s_k \geq 0$ for each k. It follows that

$$\begin{aligned} \int_{\mathbb{R}} F(x)dx &= \int_{\mathbb{R}} \sum_{n=1}^{\infty} 2^{-n} f(x - r_n)dx \\ &= \sum_{n=1}^{\infty} \int_{\mathbb{R}} 2^{-n} f(x - r_n)dx \qquad \text{(by the Monotone Convergence Theorem)} \\ &= \sum_{n=1}^{\infty} 2^{-n} \int_{\mathbb{R}} f(x)dx \\ &= \sum_{n=1}^{\infty} 2^{-n} \cdot 2 \\ &= 2 \end{aligned}$$

Hence, F is integrable. It follows directly that F converges for almost every $x \in \mathbb{R}$ (if there was a set of positive measure on which F did not converge, then the integral of F over that set would be infinite).

Let $\tilde{F}$ be as described. Since $\mathbb{Q}$ is dense in $\mathbb{R}$, any interval of $\mathbb{R}$ contains a rational number r_N a real number x_N such that, for any $\epsilon > 0$, $|x_N - r_N| < \epsilon$. Hence, for any $x \in B(r_N, \epsilon)$,

$$\begin{aligned} \tilde{F}(x) &= \sum_{n=1}^{\infty} 2^{-n} f(x - r_n)dx \\ &> 2^{-N} f(x - r_N) \\ &= 2^{-N} \epsilon^{\frac{-1}{2}} \end{aligned}$$

which is unbounded as $\epsilon \to 0$. □

Problem 2

a. Let $f \in L^r \cap L^\infty$ for some $r < \infty$. Prove that

$$||f||_p \leq ||f||_r^{\frac{r}{p}} \, ||f||_\infty^{1-\frac{r}{p}}$$

for all $r < p < \infty$.

Proof. Since $f \in L^\infty$, $|f| \leq M$ for some M. Now,

$$\begin{aligned}
\int_E |f|^p \, dx &= \int |f|^{p-r}|f|^r dx \\
&\leq \int M^{p-r}|f|^r dx \\
&= M^{p-r} \int |f|^r dx
\end{aligned}$$

which is finite, since $f \in L^r$. Hence,

$$\begin{aligned}
||f||_p &= \left(\int_E |f|^p \, dx \right)^{\frac{1}{p}} \\
&\leq (M^{p-r} \int |f|^r dx)^{\frac{1}{p}} \\
&= M^{1-\frac{r}{p}} (\int |f|^r dx)^{\frac{1}{p}} \\
&= ||f||_\infty^{1-\frac{r}{p}} \, ||f||_r^{\frac{r}{p}}
\end{aligned}$$

□

b. Assume $f \in L^r \cap L^\infty$ for some $r < \infty$. Prove

$$\lim_{p \to \infty} \|f\|_p = \|f\|_\infty$$

Proof. Observe from part (a) that

$$\begin{aligned}
||f||_p &\leq ||f||_r^{\frac{r}{p}} \, ||f||_\infty^{1-\frac{r}{p}} \\
\overline{\lim} \, ||f||_p &\leq \overline{\lim} \, ||f||_r^{\frac{r}{p}} \, ||f||_\infty^{1-\frac{r}{p}} \\
&\leq ||f||_\infty
\end{aligned}$$

Now, for $0 < t < ||f||_\infty$, define the set $A = \{x : |f(x)| \geq t\}$. Observe that, for all t, A has positive measure. Suppose this is not the case. We can find $t_0 < ||f||_\infty$ with $|f(x)| < t_0$ almost everywhere, which is a contradiction with the definition of $||f||_\infty$. Furthermore, we see that, for all t, A has finite measure. Suppose this is not the case. We see that

$$\begin{aligned}
\int |f|^r dx &\geq \int_A |f|^r dx \\
&\geq \int_A t^r dx \\
&= t^r m(A) \\
&= \infty
\end{aligned}$$

which is a contradiction with the fact that $f \in L^r$. Now, observe that $|f(x)| \geq t\chi_A(x)$ for all x, which implies that $||f||_p \geq ||t\chi_A(x)||_p$. It follows that, for any t

$$\begin{aligned}
\underline{\lim}\, ||f||_p &\geq \underline{\lim}\, ||t\chi_A||_p \\
&= \underline{\lim} \left(\int_A |t\chi_A(x)|^p dx \right)^{\frac{1}{p}} \\
&= \underline{\lim}\, (t^p m(A))^{\frac{1}{p}} \\
&= \underline{\lim}\, t m(A)^{\frac{1}{p}} \\
&= t
\end{aligned}$$

Since t can be chosen arbitrarily close to $||f||_\infty$, it follows that $\underline{\lim}\, ||f||_p \geq ||f||_\infty$. Combining this with the above, we conclude that $\lim_{p\to\infty} ||f||_p = ||f||_\infty$. □

Problem 3

Let $f_n, f \in L^p$ with $1 \leq p < \infty$. Assume $f_n(x) \to f(x)$ a.e. Prove $f_n \to f$ in L^p if and only if $\|f_n\|_p \to \|f\|_p$.

Proof. Define the function $g_n = |f|^p + |f_n|^p - |f - f_n|^p$. We have, for each n, g_n is measurable and $g_n(x) \to 2|f(x)|^p$, so $g_n \geq 0$ for all sufficiently large n. By Fatou's Lemma

$$\begin{aligned}
\int 2|f|^p dx &\leq \underline{\lim} \int g_n dx \\
&= \underline{\lim} \int |f|^p + |f_n|^p - |f - f_n|^p dx \\
&= \underline{\lim}(\int |f|^p dx + \int |f_n|^p dx - \int |f - f_n| dx) \\
&= 2 \int |f|^p dx + \underline{\lim} - \int |f - f_n|^p dx \qquad \text{(by a previous homework)} \\
&= 2 \int |f|^p dx - \overline{\lim} \int |f - f_n|^p dx
\end{aligned}$$

Hence

$$\begin{aligned}
\int 2|f|^p dx &\leq \int 2|f|^p - \overline{\lim} \int |f - f_n|^p dx \\
\overline{\lim} \int |f - f_n|^p dx &\leq 0
\end{aligned}$$

and so $\lim \int |f - f_n|^p dx = 0$. Therefore, $||f||_p \to ||f_n||_p$. □

Problem 4

Let $f \in L^1$. Denote by f_h the function $f_h(x) = f(x - h)$. Prove that $\|f - f_h\|_1 \to 0$ as $h \to 0$.

Proof. Let F be a continuous function with compact support E. We see immediately that $m(E) < \infty$. Let $\epsilon > 0$ be given. Since F is continuous, we can find $\delta > 0$ so that for $|x - y| < \delta$, $|F(x) - F(y)| < \epsilon$. Now, let $0 < h < \delta$. Since $|x - (x - h)| = h < \delta$, we conclude that $|F(x) - F(x - h)| < \epsilon$. Hence,

$$\begin{aligned}
||F - F_h||_1 &= \int |F - F_h| dx \\
&= \int_E |F - F_h| dx \\
&\leq \int_E \epsilon dx \\
&= \epsilon m(E)
\end{aligned}$$

Since ϵ is arbitrary and $m(E) < \infty$, we conclude that $||F - F_h||_1 \to 0$.

Since continuous functions with compact support are dense in L^1, for any $\epsilon > 0$, we can find continuous F with compact support such that $||F - f||_1 < \frac{\epsilon}{3}$. It follows that

$$\begin{aligned}
||f - f_h||_1 &= ||f - F + F - F_h + F_h - f_h||_1 \\
&\leq ||f - F||_1 + ||F - F_h||_1 + ||F_h - f_h||_1 \\
&= ||f - F||_1 + ||F - F_h||_1 + ||(F - f)_h||_1 \\
&= ||f - F||_1 + ||F - F_h||_1 + ||F - f||_1 \\
&< \frac{\epsilon}{3} + \frac{\epsilon}{3} + \frac{\epsilon}{3} \\
&= \epsilon
\end{aligned}$$

noting that the second term can be bounded in this way because F is continuous with compact support. $\square$