

# Math 701 Homework, Edition 8

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PROBLEM 29.

Let  $\mathbf{R}$  be a commutative ring and let  $n$  be a positive integer. Let  $J, I_0, I_1, \dots, I_{n-1}$  be ideals of  $\mathbf{R}$  so that  $I_k$  is a prime ideal for every  $k < n$  and so that  $J \subseteq I_0 \cup \dots \cup I_{n-1}$ . Prove that  $J \subseteq I_k$  for some  $k < n$ .

*Proof.* When  $n = 1$ , the claim is trivial. Suppose it holds for  $n = k$ . That is, if  $J$  is contained in a the union of  $k$  prime ideals, then  $J$  is contained in one of them. Now, for  $n = k + 1$ ,  $J \subseteq I_0 \cup \dots \cup I_k$ . Suppose that each  $I_i$  is necessary. That is, for each  $i$ , we can find  $a_i \in J$  with the property that  $a_i \in I_i$  but  $a_i \notin \bigcup_{j \neq i} I_j$ . If this were not the case (that is, no such element exists for some  $I_i$ ) then  $J \subseteq I_0 \cup \dots \cup I_{i-1} \cup I_{i+1} \cup \dots \cup I_k$ , and we are done after an appeal to the inductive hypothesis. Now, consider the sum  $\sum_j \prod_{i \neq j} a_i$ . This sum belongs to  $J$  (since  $J$  is an ideal), so it must belong to the union of the prime ideals. Hence, it belongs to at least one of the prime ideals. Without loss of generality, suppose it is  $I_0$ . Since  $a_0 \in I_0$  and  $I_0$  is an ideal, all terms containing  $a_0$  belong to  $I_0$ . By subtraction, we conclude that  $a_1 \cdots a_k \in I_0$ . Since  $I_0$  is prime, we have that one of the  $a_i$  belongs to  $I_0$ , which is a contradiction. Therefore, it must be that one of the  $I_i$  is not necessary, and so  $J$  is contained in the union of  $k$  prime ideals. By the inductive hypothesis, we conclude that  $J$  is contained in one of these ideas.  $\square$

PROBLEM 30.

Let  $\mathbf{R}$  be a nontrivial commutative ring and let  $J$  be the intersection of all the maximal proper ideals of  $\mathbf{R}$ . Prove that  $1 + a$  is a unit of  $\mathbf{R}$  for all  $a \in J$ .

*Proof.* Let  $a \in J$ . Then  $a \in M$  for each  $M$ . Since each  $M$  is a proper maximal ideal, we have  $1 \notin M$ , and in particular,  $1+a \notin M$ . As each  $M$  is maximal,  $R/M$  is a field. Since  $1+a \notin M$ , we see that  $1+a + M \neq 0 + M$ . Now,  $R/M$  is a field, so there exists  $b_M + M \in R/M$  such

that  $(1+a+M)(b_M+M) = 1+M$ . Then  $(1+a)b_M+M = 1+M \Rightarrow (1+a)b_M-1 \in 0+M$ ; in other words,  $(1+a)$  is a unit of  $R$ .  $\square$

**PROBLEM 31.**

Let  $\mathbf{F}$  be a field and let  $p(x) \in \mathbf{F}[x]$  be a polynomial of degree  $n$ . Prove that  $p(x)$  has at most  $n$  distinct roots in  $\mathbf{F}$ .

*Proof.* We proceed by induction on the degree  $n$ . Base step: if  $n = 1$ , then let  $p(x) = ax + b$  for some  $a, b \in \mathbf{F}$ ,  $a \neq 0$ . Let  $p(x) = 0$ , we have  $ax = -b$ . Since  $\mathbf{F}$  is a field, we can multiply both sides of the equation by the inverse of  $a$ . It follows that  $x = -\frac{b}{a} \in \mathbf{F}$  is a root of  $p(x)$ . Induction step: Assume  $p(x)$  has at most  $n$  roots for  $n \leq m$ . Now, consider  $n = m + 1$ , let  $c \in \mathbf{F}$  such that  $p(c) = 0$ . Then, by the factor theorem,  $p(x) = q(x)(x - c)$  for some  $q(x) \in \mathbf{F}[x]$  with  $\deg q(x) \leq m$ . By the induction hypothesis,  $q(x)$  has at most  $m$  distinct roots. Therefore,  $p(x)$  has at most  $m + 1 = n$  distinct roots.  $\square$