Math 731 Homework 8 (Correction 1)

Austin Mohr

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1 Problem 22F1

Lemma 1.1. *Let x and y be real numbers both strictly greater than -1. If $x \leq y$, then $\frac{x}{1+x} \leq \frac{y}{1+y}$.*

Proof. We have that

$$\begin{aligned} x &\leq y \\ x + xy &\leq y + xy \\ x(1+y) &\leq y(1+x) \\ \frac{x}{1+x} &\leq \frac{y}{1+y} && \text{(since } x, y \text{ strictly greater than } -1\text{).} \end{aligned}$$

□

Remark 1.2. *One can think of the previous lemma as follows: Suppose we x mL of salt and y mL of salt and we add them each to separate 1 mL containers of water. The ratio $\frac{x}{1+x}$ measures the concentration of salt in the first container and $\frac{y}{1+y}$ measures the concentration of salt in the second container. Since x is a smaller amount of salt than y, then surely the first container is less concentrated than the second. That is, $\frac{x}{1+x} \leq \frac{y}{1+y}$.*

Proposition 1.3. *If ρ is a metric on X, then both*

$$\rho_1(x, y) = \min\{1, \rho(x, y)\}$$

and

$$\rho_2(x, y) = \frac{\rho(x, y)}{1 + \rho(x, y)}$$

are metrics equivalent to ρ on X.

Proof. We first establish that ρ_1 and ρ_2 are indeed metrics on X. In what follows, let x, y, z be arbitrary points in X.

For the positive definiteness of ρ_1, observe that

$$\begin{aligned} \rho_1(x,y) &= \min\{1, \rho(x,y)\} \\ &\geq 0 && \text{(since } \rho \text{ is positive definite),} \end{aligned}$$

with equality if and only if $\rho(x,y) = 0$, which occurs if and only if $x = y$ (since ρ is positive definite).

For the symmetry of ρ_1, observe that

$$\begin{aligned} \rho_1(x,y) &= \min\{1, \rho(x,y)\} \\ &= \min\{1, \rho(y,x)\} && \text{(since } \rho \text{ is symmetric)} \\ &= \rho_1(y,x). \end{aligned}$$

For the subadditivity of ρ_1, observe that

$$\begin{aligned} \rho_1(x,y) &= \min\{1, \rho(x,y)\} \\ &\leq \min\{1, \rho(x,z) + \rho(z,y)\} && \text{(since } \rho \text{ is subadditive)} \\ &\leq \min\{1, \rho(x,z)\} + \min\{1, \rho(z,y)\} \\ &= \rho_1(x,z) + \rho(z,y). \end{aligned}$$

Therefore, ρ_1 is a metric on X.

For the positive definiteness of ρ_2, observe that

$$\begin{aligned} \rho_2(x,y) &= \frac{\rho(x,y)}{1 + \rho(x,y)} \\ &\geq 0 && \text{(since } \rho \text{ is positive definite),} \end{aligned}$$

with equality if and only if $\rho(x,y) = 0$, which occurs if and only if $x = y$ (since ρ is positive definite).

For the symmetry of ρ_2, observe that

$$\begin{aligned} \rho_2(x,y) &= \frac{\rho(x,y)}{1 + \rho(x,y)} \\ &= \frac{\rho(y,x)}{1 + \rho(y,x)} && \text{(since } \rho \text{ is symmetric)} \\ &= \rho_2(y,x). \end{aligned}$$

For the subadditivity of ρ_2, observe that

$$\begin{aligned} \rho(x,y) &\leq \rho(x,z) + \rho(z,y) && \text{(since } \rho \text{ is subadditive)} \\ \frac{\rho(x,y)}{1+\rho(x,y)} &\leq \frac{\rho(x,z)+\rho(z,y)}{1+\rho(x,z)+\rho(z,y)} && \text{(by lemma).} \end{aligned}$$

The lefthand side of the inequality is precisely $\rho_2(x,y)$, and so we have that

$$\begin{aligned} \rho_2(x,y) &\leq \frac{\rho(x,z)+\rho(z,y)}{1+\rho(x,z)+\rho(z,y)} \\ &= \frac{\rho(x,z)}{1+\rho(x,z)+\rho(z,y)} + \frac{\rho(z,y)}{1+\rho(x,z)+\rho(z,y)} \\ &\leq \frac{\rho(x,z)}{1+\rho(x,z)} + \frac{\rho(z,y)}{1+\rho(z,y)} && \text{(since } \rho \text{ is positive definite)} \\ &= \rho_2(x,z) + \rho_2(z,y). \end{aligned}$$

Therefore, ρ_2 is a metric on X.

We show next that ρ and ρ_1 are equivalent by showing (X,ρ) and (X,ρ_1) have the same open sets. To that end, consider the basic open set $B_\rho(x,\epsilon)$ of (X,ρ). We have that

$$B_\rho(x,\epsilon) = \bigcup_{y \in B_\rho(x,\epsilon)} B_\rho(y,\epsilon_y),$$

where each ϵ_y depends on y. By definition of ρ_1, we have, for each y,

$$B_{\rho_1}(y,\epsilon_y) \subset B_\rho(y,\epsilon_y).$$

Taken together, we see that

$$B_\rho(x,\epsilon) = \bigcup_{y \in B_\rho(x,\epsilon)} B_{\rho_1}(y,\epsilon_y),$$

and so $(X,\rho) \subset (X,\rho_1)$.

Next, consider the basic open set $B_{\rho_1}(x,\epsilon)$ of (X,ρ_1). As before,

$$B_{\rho_1}(x,\epsilon) = \bigcup_{y \in B_{\rho_1}(x,\epsilon)} B_{\rho_1}(y,\epsilon_y)$$

where each ϵ_y depends on y. In particular, we may insist that $0 < \epsilon_y \leq 1$ for all y. With this restriction, we have, for each y,

$$B_{\rho_1}(y, \epsilon_y) = B_\rho(y, \epsilon_y).$$

Taken together, we see that

$$B_{\rho_1}(x, \epsilon) = \bigcup_{y \in B_{\rho_1}(x,\epsilon)} B_\rho(y, \epsilon_y),$$

and so $(X, \rho_1) \subset (X, \rho)$.

We show next that ρ and ρ_2 are equivalent by showing that (X, ρ) and (X, ρ_2) have the same open sets. That $(X, \rho) \subset (X, \rho_2)$ follows precisely as in the proof that $(X, \rho) \subset (X, \rho_1)$, as $B_{\rho_2}(x, \epsilon) \subset B_\rho(x, \epsilon)$ for all $x \in X$ and $\epsilon > 0$.

Next, consider the basic open set $B_{\rho_2}(x, \epsilon)$ of (X, ρ_2). As before,

$$B_{\rho_2}(x, \epsilon) = \bigcup_{y \in B_{\rho_2}(x,\epsilon)} B_{\rho_2}(y, \epsilon_y),$$

where each ϵ_y depends on y. Observe now that, for all y,

$$\begin{aligned} \rho_2(y, z) = \epsilon_y &\Rightarrow \frac{\rho(y, z)}{1 + \rho(y, z)} = \epsilon_y \\ &\Rightarrow \rho(y, z) = \epsilon_y(1 + \rho(y, z)) \\ &\Rightarrow \rho(y, z) = \frac{\epsilon_y}{1 - \epsilon_y}, \end{aligned}$$

where we insist that $0 < \epsilon_y < 1$ for all y. Taken together, we see that

$$B_{\rho_2}(x, \epsilon) = \bigcup_{y \in B_{\rho_2}(x,\epsilon)} B_\rho(y, \frac{\epsilon_y}{1 - \epsilon_y}),$$

and so $(X, \rho_2) \subset (X, \rho)$. □

2 Problem 22F2

Proposition 2.1. *Every metric generating the topology of a compact metrizable space is bounded.*

Proof. We proceed by establishing the contrapositive. To that end, let ρ be an unbounded metric generating a topology on a set X. Define, for all $n \in \mathbb{N}$, B_n to be the open set $B(x, n)$ for some fixed $x \in X$. The collection $\{B_n \mid n \in \mathbb{N}\}$ covers X but admits no finite subcover. To see this, let $\mathcal{C}$ be any finite subcollection of $\{B_n \mid n \in \mathbb{N}\}$. By construction, $\bigcup \mathcal{C} \subset B(x, M)$, where M denotes the maximum n such that $B_n \in \mathcal{C}$. Since ρ is unbounded, there is a point $y \in X$ with $\rho(x, y) > M$, and so $y \notin B(x, M)$. Hence, $\mathcal{C}$ does not cover X, and so X is not compact, thus establishing the contrapositive. $\square$