Austin Mohr
Math 704
Homework 7

Problem 1
a. For $m(E) < \infty$, show that

$$L^\infty(E) \subset L^r(E) \subset L^p(E) \subset L^1(E)$$

where $1 < p < r < \infty$. Show, for $E = (0, 1]$, by example that all the inclusions can be strict.

Claim. $L^\infty(E) \subset L^r(E)$

Proof. Let $f \in L^\infty(E)$. We have that f is measurable and, for some M, $|f(x)| \leq M$ almost everywhere on E. Hence

$$\begin{aligned} |f(x)|^r &\leq M^r \\ \int_E |f(x)|^r dx &\leq M^r m(E) \\ &< \infty \end{aligned}$$

Hence, $f \in L^r(E)$.

To see that this inclusion is strict, consider the function $f = \frac{1}{x^{\frac{1}{2r}}}$ on $E = (0, 1]$. We see that f is unbounded, so $f \notin L^\infty(E)$, but

$$\begin{aligned} \int_E |f(x)|^r \, dx &= \int_E \left| \frac{1}{x^{\frac{1}{2r}}} \right|^r dx \\ &= \int_E \left| \frac{1}{x^{\frac{1}{2}}} \right| dx \\ &= 2 \\ &< \infty \end{aligned}$$

so $f \in L^r$. □

Claim. $L^r(E) \subset L^p(E)$

Proof. Let $f \in L^r(E)$. We have that f is measurable and $\left(\int_E |f(x)|^r dx\right)^{\frac{1}{r}} < \infty$. Now, let $n = \frac{r}{p}$. Observe that $\frac{1}{n} = \frac{p}{r} < 1$, so there exists a number q so that $\frac{1}{n} + \frac{1}{q} = 1$. Let $g(x) = (f(x))^p$ for all x. We show that $g \in L^n(E)$. Since $f \in L^r(E)$, f is measurable, and so $g = f^p$ is measurable. Furthermore,

$$\begin{aligned} \int_E |g|^n dx &= \int_E |(f(x))^p|^{\frac{r}{p}} \, dx \\ &= \int_E |f(x)|^r dx \\ &< \infty \end{aligned}$$

We also see that the constant function 1 is in $L^q(E)$, since 1 is measurable and

$$\begin{aligned} \int_E |1|^q dx &= \int_E dx \\ &= m(E) \\ &< \infty \end{aligned}$$

Next, apply Hölder's Inequality to $g \cdot 1$ to obtain

$$\begin{aligned}\int_E |g(x) \cdot 1| dx &\leq ||g||_n \, ||1||_q \\ &= \left(\int_E |g(x)|^n \, dx \right)^{\frac{1}{n}} \left(\int_E |1|^q \, dx \right)^{\frac{1}{q}} \\ &< \infty\end{aligned}$$

Since $\int_E |g(x) \cdot 1| dx = \int_E |f(x)|^p dx$, we have that $f \in L^p$.

To see that this inclusion is strict, consider the function $f = \frac{1}{x^{\frac{1}{r}}}$ on $E = (0, 1]$. Observe that f is measurable. Now, we see that $f \notin L^r(E)$, since

$$\begin{aligned}\int_E |f|^r \, dx &= \int_E \left| \frac{1}{x^{\frac{1}{r}}} \right|^r dx \\ &= \int_E \left| \frac{1}{x} \right|^r dx \\ &= \infty\end{aligned}$$

Now,

$$\begin{aligned}\int_E |f|^p \, dx &= \int_E \left| \frac{1}{x^{\frac{1}{r}}} \right|^p dx \\ &= \int_E \left| \frac{1}{x^{\frac{p}{r}}} \right| dx \\ &< \infty\end{aligned}$$

since $\frac{p}{r} < 1$. Hence, $f \in L^p(E)$. □

Claim. $L^p \subset L^1$

Proof. As in the previous proof, but replace every occurrence of "r" with "p" and every occurrence of "p" with "1". Similarly, this inclusion is strict. □

b. Show that in general (i.e., if $m(E) = \infty$)

$$L^\infty \cap L^1 \subset L^p \subset L^\infty + L^1 = \{f : f = g + h, g \in L^\infty, h \in L^1\}$$

Claim. $L^\infty \cap L^1 \subset L^p$

Proof. Let $f \in L^\infty \cap L^1$. We know that $|f|$ is bounded (say by M) and integrable. It follows that

$$\begin{aligned}\int_E |f|^p dx &= \int_E |f||f|^{p-1} dx \\ &\leq M^{p-1} \int_E |f| dx \\ &< \infty\end{aligned}$$

Therefore, $f \in L^p$. □

Claim. $L^p \subset L^\infty + L^1$

Proof. Let $f \in L^p$. We know that $\int_E |f|^p dx < \infty$. Observe that

$$E = \{x : |f(x)|^p < 1\} \cup \{x : |f(x)|^p \geq 1\}$$

Define the function g to be f restricted to $\{x : |f(x)|^p < 1\}$ and the function h to be f restricted to $\{x : |f(x)|^p \geq 1\}$. First, observe that that $f = g + h$ with g and h both measurable. Next, we see that $g \in L^\infty$, since $|g(x)| < 1$ for all x in its domain. Finally, if we denote $\{x : |f(x)|^p \geq 1\}$ by H,

$$\begin{aligned}\int_H |h(x)|dx &= \int_H |f(x)|dx \\ &\leq \int_H |f(x)|^p dx \\ &\leq \int_E |f(x)|^p dx \\ &< \infty\end{aligned}$$

Hence, $h \in L^1$. □

Problem 2

Let $f \in L^2([0,1])$. Prove that

$$\left(\int_{[0,1]} xf(x)dx\right)^2 \leq \tfrac{1}{3}\int_{[0,1]} |f(x)|^2 dx$$

Proof. Observe first that the function x is in $L^2([0,1])$, since x is measurable and

$$\begin{aligned}\int_{[0,1]} |x|^2\, dx &= \int_{[0,1]} x^2 dx \\ &= \frac{1}{3} \\ &< \infty\end{aligned}$$

□

Now,

$$\begin{aligned}\left(\int_{[0,1]} xf(x)dx\right)^2 &= \left|\int_{[0,1]} xf(x)dx\right|^2 \\ &\leq \left(\int_{[0,1]} |xf(x)|dx\right)^2 \\ &\leq \left(||x||_2\, ||f(x)||_2\right)^2 \qquad \text{(by Hölder's Inequality)} \\ &= \left(\left(\int_E |x|^2\, dx\right)^{\frac{1}{2}} \left(\int_E |f(x)|^2\, dx\right)^{\frac{1}{2}}\right)^2 \\ &= \int_{[0,1]} |x|^2 dx \int_{[0,1]} |f(x)|^2 dx \\ &= \frac{1}{3}\int_{[0,1]} |f(x)|^2 dx\end{aligned}$$

Problem 3

Let E be a measurable set of finite measure and let $1 < p < \infty$. Assume $f_n \in L^p(E)$ such that $||f_n||_p \leq 1$ and $f_n(x) \to 0$ almost everywhere. Prove that $||f_n||_1 \to 0$.

Proof. By Egorov's Theorem, we can find, for any $\epsilon_0 > 0$, a subset A_{ϵ_0} of E with $m(E \setminus A_{\epsilon_0}) < \epsilon_0$ so that $f_n \to 0$ uniformly on A_{ϵ_0}. This implies that, for any $\epsilon > 0$, there is an N so that for all $n \geq N$,

$$\begin{aligned}\int_{A_{\epsilon_0}} |f_n| dx &\leq \int_{A_{\epsilon_0}} \epsilon dx \\ &= \epsilon m(A_{\epsilon_0}) \\ &< \epsilon m(E)\end{aligned}$$

Hence, $\int_{A_{\epsilon_0}} |f_n| dx \to 0$.

For the remainder of the domain, observe that

$$\int_{E\setminus A_{\epsilon_0}} |f_n| dx = \int_E |f_n \chi_{E\setminus A_{\epsilon_0}}| dx$$

Since $1 < p < \infty$, we can choose q so that $\frac{1}{p} + \frac{1}{q} = 1$. We already know that $f_n \in L^p(E)$ for all n. In order to apply Hölder's Inequality, we need that $\chi_{E\setminus A_{\epsilon_0}} \in L^q(E)$. This is true since

$$\begin{aligned}\int_E |\chi_{E\setminus A_{\epsilon_0}}|^q dx &< \int_E |1|^q dx \\ &= \int_E dx \\ &= m(E) \\ &< \infty\end{aligned}$$

Applying Holder's Inequality, we see that

$$\begin{aligned}\int_{E\setminus A_{\epsilon_0}} |f_n| dx &= \int_E |f_n \chi_{E\setminus A_{\epsilon_0}}| dx \\ &\leq ||f_n||_p \left|\left|\chi_{E\setminus A_{\epsilon_0}}\right|\right|_q \\ &\leq \left|\left|\chi_{E\setminus A_{\epsilon_0}}\right|\right|_q \\ &= \left(\int_E \left|\chi_{E\setminus A_{\epsilon_0}}\right|^q dx\right)^{\frac{1}{q}} \\ &= \left(\int_E \chi_{E\setminus A_{\epsilon_0}} dx\right)^{\frac{1}{q}} \\ &= (m(E \setminus A_{\epsilon_0}))^{\frac{1}{q}}\end{aligned}$$

Since $m(E \setminus A_{\epsilon_0})$ can be made arbitrarily small, we conclude that $\int_{E\setminus A_{\epsilon_0}} |f_n| dx \to 0$. Taken together with the fact that $\int_{A_{\epsilon_0}} |f_n| dx \to 0$, we have that $||f_n||_1 \to 0$. □

Problem 4

Let $f_n \to f$ in L^p, $1 \le p < \infty$, and let $\{g_n\}$ be a sequence of measurable functions such that $|g_n| \le M$ for all n with $g_n \to g$ almost everywhere.

a. Prove $||(g_n - g)f||_p \to 0$.

Proof. Observe first that

$$\begin{aligned} ||(g_n - g)f||_p &= \left(\int_E |(g_n - g)f|^p \, dx\right)^{\frac{1}{p}} \\ &= \left(\int |g_n - g|^p |f|^p dx\right)^{\frac{1}{p}} \end{aligned}$$

The proof proceeds by establishing the hypotheses for the Dominated Convergence Theorem. Define $F_n = |g_n - g|^p |f|^p$. Now, $f \in L^p$ implies that f is finite almost everywhere. Taken with the fact that $g_n \to g$ almost everywhere, we have that $F_n \to 0$ almost everywhere. Next, observe that, since $|g_n| \le M$ and $g_n \to g$ almost everywhere, $|g_n - g| \le 2M$ almost everywhere. Define the function $G = (2M)^p |f|^p$. We see that $|F_n| \le G$ almost everywhere and G is integrable (since $f \in L^p$). It follows that

$$\begin{aligned} \lim_{n\to\infty} \int |g_n - g|^p |f|^p dx &= \lim_{n\to\infty} \int F_n dx \\ &= \int \lim_{n\to\infty} F_n dx \qquad \text{(by the Dominated Convergence Theorem)} \\ &= 0 \end{aligned}$$

Therefore, $||(g_n - g)f||_p \to 0$. □

b. Prove $g_n f_n \to fg$ in L^p.

Proof.

$$\begin{aligned} ||g_n f_n - fg||_p &= ||g_n f_n - fg + g_n f - g_n f||_p \\ &= ||(g_n - g)f + (f_n - f)g_n||_p \\ &\le ||(g_n - g)f||_p + ||(f_n - f)g_n||_p \\ &\le ||(g_n - g)f||_p + ||(f_n - f)M||_p \\ &= ||(g_n - g)f||_p + M\,||f_n - f||_p \end{aligned}$$

The first term goes to 0 by part (a) and the second term goes to 0 by the assumption that $f_n \to f$ in L^p. Hence, $g_n f_n \to fg$ in L^p. □