

Math 701 Homework, Edition 7

Kenny Brown, Brian Hanson, Jing Liu
Austin Mohr, Pam Welch

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PROBLEM 25.

(1) Let I and J be ideals of a commutative ring \mathbf{R} with $I + J = R$. Prove that $IJ = I \cap J$.

(2) Let I , J , and K be ideals of a principal ideal domain. Prove that $I \cap (J + K) = I \cap J + I \cap K$

Proof. 1) Let $x \in IJ$. Then $x = x_i + x_j$ where $x_i \in I$ and $x_j \in J$. Therefore, since I and J are ideals, by definition, $x_i x_j \in I$ and $x_i x_j \in J$. Thus, $IJ \subseteq I \cap J$.

Let $x \in I \cap J$. Since $I + J = R$, then $1 = i + j$ where $i \in I$ and $j \in J$. Then $x = 1x = (i + j)x = ix + jx$ and therefore, $x \in IJ$. So, $I \cap J \subseteq IJ$ and $IJ = I \cap J$. \square

PROBLEM 26.

Let R be a commutative ring and I be a proper prime ideal of R such that R/I satisfies the descending chain condition on ideals. Prove that R/I is a field.

Proof. Since I is a proper prime ideal of R , R/I is an integral domain. Now, pick $a \in R/I$ with $a \neq 0$, and consider the ideal generated by a . Since R/I satisfies the descending chain condition, we have

$$(a) \supseteq (a^2) \supseteq (a^3) \supset \cdots \supset (a^k) = (a^{k+1}) = \cdots$$

So we have $a^k = ba^{k+1}$ for some b . Now, since R/I is an integral domain, we see that $ba^{k+1} - a^k = 0$ and so $ba = 1$. Thus $b = a^{-1}$, and so R/I is a field. \square

PROBLEM 27.

Let \mathbf{R} be a commutative ring and I be an ideal which is contained in a prime ideal P . Prove that the collection of prime ideals contained in P and containing I has a minimal member.

Proof. Let F be the set of all prime ideals that contain I and are contained in P . Let C be chain of prime ideals in F ordered by \supseteq . Observe that $\cap C$ is an upper bound. Furthermore, $I \subseteq \cap C \subseteq P$. We claim that $\cap C$ is indeed a prime ideal of \mathbf{R} .

We see that 0 is an element of every prime ideal in the chain, so 0 is an element of $\cap C$.

Let a and b be elements of $\cap C$. We have that a and b belong to every prime ideal in the chain, so $a + b$ and $-a$ belong to every prime ideal in the chain. Hence, $a + b$ and $-a$ belong to $\cap C$.

Let a be an element of $\cap C$ and r an element of the ring \mathbf{R} . We see that a belongs to every prime ideal in the chain, so ra and ar belong to every prime ideal in the chain. Hence, ra and ar belong to $\cap C$.

Let ab be an element of $\cap C$. We see that ab belongs to every prime ideal in the chain. Hence, either a or b (or both) belongs to every prime ideal in the chain. Since the chain is ordered by \supseteq , every prime ideal contains a or every prime ideal contains b (or both). In other words, we cannot have prime ideals P_i and P_j with $P_i \supseteq P_j$, $a \in P_i \not\supseteq b$, and $a \notin P_j \supseteq b$. Hence, either a or b (or both) belongs to $\cap C$.

The preceding observations imply that $\cap C$ is indeed a prime ideal of \mathbf{R} . By Zorn's Lemma, we conclude that F has a minimal element (i.e. a maximal element with respect to \supseteq). \square

PROBLEM 28.

Let X be a finite set and let \mathbf{R} be the ring of functions from X into the field \mathbb{R} of real numbers. Prove that an ideal M of \mathbf{R} is maximal if and only if there is an element $a \in X$ such that $M = \{f \mid f \in R \text{ and } f(a) = 0\}$.

Proof. To prove the first direction, let $X = \{x_1, x_2, \dots, x_n\}$, $R = \{f \mid f : X \rightarrow \mathbb{R}\}$. Without loss of generality, let $a = x_1 \in X$, let $M = \{f \mid f \in R \text{ and } f(x_1) = 0\}$. Now, we show that M is a maximal ideal. Clearly, M is a proper ideal of R . Let I be an ideal of R such that $M \subseteq I$ and $M \neq I$. Then there exists $f_1 \in I$, $f_1 \notin M$. So, $f_1(x_1) \neq 0$. Now, define f_i for $2 \leq i \leq n$, $1 \leq j \leq n$ by

$$f_i(x_j) = \begin{cases} 1 & \text{if } j = i \\ 0 & \text{if } j \neq i \end{cases}$$

Then, $f_i(x_1) = 0$ for $i = 2, \dots, n$. So, $f_i(x) \in M \subseteq I$ for $i = 2, \dots, n$. Now, let

$$h(x) := \frac{f_1(x)}{f_1(x_1)} + f_2(x) + \dots + f_n(x) \in I$$

Notice that $h(x) = 1$ for all $x_i \in X$. Therefore $I = R$. This means that M is an maximal ideal of R .

To prove the other direction, Let M be a maximal ideal of R . Assume that for all $x_i \in X$, there exists $g_i \in M$, such that $g_i(x_i) \neq 0$. Now, define $h_i(x)$ for $i = 1, \dots, n$ and $j = 1, \dots, n$ by

$$h_i(x_j) = \begin{cases} \frac{1}{g_i(x_i)} & \text{if } j = i \\ 0 & \text{if } j \neq i \end{cases}$$

Then, $h_i(x)g_i(x) \in M$ for $i = 1, \dots, n$. Let

$$\phi(x) := h_1(x)g_1(x) + h_2(x)g_2(x) + \dots + h_n(x)g_n(x) \in M.$$

Also, notice that $\phi(x) \equiv 1$. This means that $M = R$. This is a contradiction. Therefore our assumption must be false. So $M \subseteq \{f \mid f \in R \text{ and } f(a) = 0\}$ for some $a \in X$. Since we already showed that the right set is a maximal ideal. So, $M = \{f \mid f \in R \text{ and } f(a) = 0\}$. \square